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Equivariant moduli problems, branched manifolds, and the Euler class ☆

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1. Introduction

The purpose of this paper is to explain the equivariant Euler class associated to an oriented G-equivariant Fredholm section $\mathscr{S} : \mathscr{B} \to \mathscr{E}$ of a Hilbert space bundle over a Hilbert manifold. The key hypotheses are that the Lie group G is compact, the isotropy subgroups are finite, and the zero set of the section is compact. The present paper is motivated by our joint work with Gaio [7] on invariants of Hamiltonian group actions. In this work, the Fredholm section arises from a version of the vortex equations, where the target space is a symplectic manifold with a Hamiltonian G-action [8,19,20]. In many interesting cases, the resulting moduli spaces are compact and so the results of the present paper can be applied. Other examples of Fredholm sections with compact zero sets are the Seiberg–Witten equations over a four-manifold [25] or the harmonic map equations when the target space is a negatively curved manifold (see e.g. [14]). This is in sharp contrast to the Gromov–Witten invariants of general (compact) symplectic manifolds [11,16,17,22] and to the Donaldson invariants of smooth four-manifolds [9], where the moduli spaces are noncompact and the compactifications are the source of some major difficulties of the theory. Since the unperturbed moduli space is compact, our framework is considerably simpler than the one required for the construction of the Gromov–Witten invariants. Our exposition follows closely the work of Li et al. [16].

In the case $G = \{\mathbb{1}\}$ similar results were proved in [6,12,21]. In [12] Fulton proved that, if B is a finite dimensional complex manifold, $E \to B$ is a holomorphic vector bundle, and $S : B \to E$ is a holomorphic section, then the zero set $M := S^{-1}(0)$ carries a fundamental cycle (in singular homology) which is Poincaré dual to the Euler class. This was extended to the infinite dimensional setting by Pidstrigatch–Tyurin [21] and to the nonholomorphic case by Brussee [6]. The last two

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references contain applications to the topology of Kähler surfaces via Donaldson and Seiberg–Witten theory. They use finite dimensional reduction (in the nonequivariant case) as we do in Section 7, and [21] contains a version of the localization result (Theorem 11.1) in the case where all the weights are one.

One can think of the "virtual fundamental class" of the zero set

$$\mathscr{M} := \mathscr{S}^{-1}(0)$$

as a homomorphism $\chi^{\mathscr{B},\mathscr{E},\mathscr{S}} : H^*_{\mathrm{G}}(\mathscr{B};\mathbb{R}) \to \mathbb{R}$ obtained by "integrating" an equivariant cohomology class $\alpha \in H^*_{\mathrm{G}}(\mathscr{B})$ over $\mathscr{M}/\mathrm{G}$:

$$\chi^{\mathscr{B},\mathscr{E},\mathscr{S}}(\alpha) := \int_{\mathscr{M}/\mathrm{G}} \alpha.$$

In the physics literature this is often described as the "integral" of the cup product of α with the "Euler class" of the bundle $\mathscr{E}$ over the infinite dimensional orbifold $\mathscr{B}/\mathrm{G}$. We shall adopt this terminology and call the homomorphism $\chi^{\mathscr{B},\mathscr{E},\mathscr{S}}$ the *Euler class* of the triple $(\mathscr{B},\mathscr{E},\mathscr{S})$. If $\mathscr{S}$ is transverse to the zero section (i.e. the vertical differential $\mathscr{D}_x := D\mathscr{S}(x) : T_x\mathscr{B} \to \mathscr{E}_x$ is surjective for every $x \in \mathscr{M}$) and G acts freely on $\mathscr{M} = \mathscr{S}^{-1}(0)$ then $\mathscr{M}/\mathrm{G}$ is an oriented smooth compact manifold and integration of α over $\mathscr{M}/\mathrm{G}$ can be understood literally. Another interesting case, first used by Mrowka in the context of Seiberg–Witten theory, is where the cokernel of $\mathscr{D}_x$ has constant rank along $\mathscr{M}$, the zero set $\mathscr{M}$ is a smooth submanifold of $\mathscr{B}$ with tangent space $T_x\mathscr{M} = \ker \mathscr{D}_x$, and G acts freely on $\mathscr{M}$. In this case, one can integrate an equivariant cohomology class on $\mathscr{B}$ by pulling it back to $\mathscr{M}/\mathrm{G}$ and taking the cup product with the Euler class of the *obstruction bundle* $\operatorname{coker} \mathscr{D}/\mathrm{G} \to \mathscr{M}/\mathrm{G}$. In the presence of nontrivial isotropy subgroups there may not exist a perturbation of $\mathscr{S}$ that is both G-equivariant and transverse to the zero section. We present two constructions to overcome this difficulty in the finite dimensional case.

The first construction follows the work of Ruan [16,22] and circumvents the transversality problem by pulling back a Thom form τ on E by the section S and integrating the product of a differential form with $S^*\tau$ over the base. The integration will be meaningful because the Thom form can be chosen such that the pullback $S^*\tau$ is supported in an arbitrarily small neighbourhood of M.

In the second construction we perturb the section S by a "multivalued section" $\sigma : B \to 2^E$. This can be done such that $S - \sigma$ is G-equivariant and transverse to the zero section. Its zero set $(S-\sigma)^{-1}(0)$ is then a "weighted branched submanifold" which represents a rational homology cycle.

Section 2 begins with a formal definition of the category of G-moduli problems and discusses the axiomatic properties of the Euler class. The remainder of the paper is devoted to the existence proof. The five subsequent sections are of preparatory nature. In Section 3 we construct an explicit isomorphism between the equivariant cohomology groups $H^*_{\mathrm{G}}(B)$ and $H^*_{\mathrm{G/H}}(B/\mathrm{H})$, where H is a normal subgroup of G. These results are useful for the construction of Thom forms and follow the work of Guillemin–Sternberg in [13]. The next three sections deal with integration of compactly supported equivariant differential forms in the presence of finite isotropy (Section 4), the construction of the equivariant Thom class (Section 5), and integration over the fibre for equivariant vector bundles (Section 6). Section 7 explains how to reduce infinite dimensional moduli problems to finite dimensional ones. In Section 8 we combine the preceding five sections to define the Euler class. In Sections 9 and 10 we develop the theory of weighted branched submanifolds. We show that multivalued perturbations give rise to weighted branched submanifolds, that the Euler class can be represented by a compact oriented weighted branched submanifold, and that every compact oriented weighted branched submanifold represents a rational homology class. Section 11 contains a localization theorem for circle actions.

2. The Euler class for G-moduli problems

We begin with a general definition of G-moduli problems in a Hilbert space setting.

Definition 2.1. Let G be a compact oriented Lie group. A G-*moduli problem* is a triple $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ with the following properties.

- $\mathscr{B}$ is a Hilbert manifold (without boundary) equipped with a smooth G-action.
- $\mathscr{E}$ is a Hilbert space bundle over $\mathscr{B}$, also equipped with a smooth G-action, such that G acts by isometries on the fibres of $\mathscr{E}$ and the projection $\mathscr{E} \to \mathscr{B}$ is G-equivariant.
- $\mathscr{S} : \mathscr{B} \to \mathscr{E}$ is a smooth G-equivariant Fredholm section of constant Fredholm index such that the determinant bundle $\det(\mathscr{S}) \to \mathscr{B}$ is oriented, G acts by orientation preserving isomorphisms on the determinant bundle, and the zero set

  $$\mathscr{M} := \{x \in \mathscr{B} \mid \mathscr{S}(x) = 0\}$$

  is compact.

A finite dimensional G-moduli problem (B, E, S) is called *oriented* if B and E are oriented and G acts on B and E by orientation preserving diffeomorphisms. A G-moduli problem $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ is called *regular* if the isotropy subgroup $\mathrm{G}_x := \{g \in \mathrm{G} \mid g^*x = x\}$ is finite for every $x \in \mathscr{M}$.

Remark 2.2. If (B, E, S) is a finite dimensional G-moduli problem, then B need not be an orientable manifold. However, it follows from the definition that the total space of the vector bundle E is an oriented manifold (or, equivalently, $TB \oplus E$ is an oriented vector bundle over B) and G acts on E by orientation preserving diffeomorphisms (or, equivalently, it acts on the fibres of $TB \oplus E$ by orientation preserving isomorphisms). If S is transverse to the zero section then the orientation of $TB \oplus E$ determines an orientation of $M = S^{-1}(0)$ and G acts on M by orientation preserving diffeomorphisms.

Example 2.3. An example of a finite dimensional G-moduli problem is given by $\mathrm{G} = \mathbb{Z}_2$, $B = \mathbb{R}$, $E = \mathbb{R} \times \mathbb{R}$, and $S(x) = x \in E_x = \mathbb{R}$, where the action of $\mathbb{Z}_2$ on E is given by $(x, y) \mapsto (-x, -y)$. In this case B and E are oriented manifolds and G acts on E by orientation preserving diffeomorphisms. But G does not act on B by orientation preserving diffeomorphisms. So (B, E, S) is not oriented in the sense of Definition 2.1.

Let $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ be a G-moduli problem. The fibre of $\mathscr{E}$ over $x \in \mathscr{B}$ will be denoted by $\mathscr{E}_x$. Thus elements of $\mathscr{E}$ are pairs (x, e), where $x \in \mathscr{B}$ and $e \in \mathscr{E}_x$. In this notation a section is a map of the form $\mathscr{B} \to \mathscr{E} : x \mapsto (x, \mathscr{S}(x))$, where $\mathscr{S}(x) \in \mathscr{E}_x$. Abusing notation, we also denote the map $\mathscr{B} \to \mathscr{E}$ by $\mathscr{S}$. The Fredholm property asserts that, for $x \in \mathscr{M} = \mathscr{S}^{-1}(0)$, the vertical differential

$$\mathscr{D}_x := D\mathscr{S}(x) : T_x\mathscr{B} \to \mathscr{E}_x$$

is a Fredholm operator whose Fredholm index is independent of x. This implies that the vertical differential of $\mathscr{S}$, with respect to any trivialization of $\mathscr{E}$, is Fredholm in a sufficiently small neighbourhood of $\mathscr{M}$. The orientation hypothesis asserts that the determinant bundle is oriented over such a neighbourhood. We define the *index* of $\mathscr{S}$ by

$$\mathrm{index}(S) := \mathrm{index}(\mathscr{D}_x) - \dim \mathrm{G}.$$

This is the index of the elliptic complex $0 \to \mathfrak{g} \to T_x\mathscr{B} \to \mathscr{E}_x \to 0$, where the map $\mathfrak{g} \to T_x\mathscr{B}$ is the infinitesimal action. G-moduli problems form a category as follows.

Definition 2.4. Let $(\mathscr{B}, \mathscr{E}, \mathscr{S})$, $(\mathscr{B}', \mathscr{E}', \mathscr{S}')$ be G-moduli problems. A *morphism* from $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ to $(\mathscr{B}', \mathscr{E}', \mathscr{S}')$ is a pair (ψ, Ψ) with the following properties:

$$\psi : \mathscr{B}_0 \to \mathscr{B}'$$

is a smooth G-equivariant embedding of a neighbourhood $\mathscr{B}_0 \subset \mathscr{B}$ of $\mathscr{M}$ into $\mathscr{B}'$,

$$\Psi : \mathscr{E}_0 := \mathscr{E}|_{\mathscr{B}_0} \to \mathscr{E}'$$

is a smooth injective bundle homomorphism and a lift of ψ, and the sections $\mathscr{S}$ and $\mathscr{S}'$ satisfy

$$\mathscr{S}' \circ \psi = \Psi \circ \mathscr{S}, \qquad \mathscr{M}' = \psi(\mathscr{M}).$$

Moreover, the linear operators $d_x\psi : T_x\mathscr{B} \to T_{\psi(x)}\mathscr{B}'$ and $\Psi_x : \mathscr{E}_x \to \mathscr{E}'_{\psi(x)}$ induce isomorphisms

$$d_x\psi : \ker \mathscr{D}_x \to \ker \mathscr{D}'_{\psi(x)}, \qquad \Psi_x : \mathrm{coker}\, \mathscr{D}_x \to \mathrm{coker}\, \mathscr{D}'_{\psi(x)} \tag{1}$$

for $x \in \mathscr{M}$, and the resulting isomorphism from $\det(\mathscr{S})$ to $\det(\mathscr{S}')$ is orientation preserving.

Let $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ and $(\mathscr{B}', \mathscr{E}', \mathscr{S}')$ be G-moduli problems and suppose that there exists a morphism from $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ to $(\mathscr{B}', \mathscr{E}', \mathscr{S}')$. Then the indices of $\mathscr{S}$ and $\mathscr{S}'$ agree. Moreover, $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ is regular if and only if $(\mathscr{B}', \mathscr{E}', \mathscr{S}')$ is regular.

Definition 2.5. Two regular G-moduli problems $(\mathscr{B}_i, \mathscr{E}_i, \mathscr{S}_i)$, $i = 0, 1$, are called *cobordant* if there exist a G-equivariant Hilbert space bundle $\tilde{\mathscr{E}} \to \tilde{\mathscr{B}}$ over a Hilbert manifold $\tilde{\mathscr{B}}$ with boundary, a smooth oriented G-equivariant Fredholm section $\tilde{\mathscr{S}} : \tilde{\mathscr{B}} \to \tilde{\mathscr{E}}$ such that the zero set $\tilde{\mathscr{M}} := \tilde{\mathscr{S}}^{-1}(0)$ is compact, G acts with finite isotropy on $\tilde{\mathscr{B}}$, and

$$\partial\tilde{\mathscr{B}} = \mathscr{B}_0 \cup \mathscr{B}_1, \qquad \mathscr{E}_i = \tilde{\mathscr{E}}|_{\mathscr{B}_i}, \qquad \mathscr{S}_i = \tilde{\mathscr{S}}|_{\mathscr{B}_i}.$$

Moreover, $\det(\tilde{\mathscr{S}})$ carries an orientation which induces the orientation of $\det(\mathscr{S}_1)$ over $\mathscr{B}_1$ and the opposite of the orientation of $\det(\mathscr{S}_0)$ over $\mathscr{B}_0$. Here an orientation of $\det(\tilde{\mathscr{S}})$ induces an orientation of the determinant bundle of $\mathscr{S} := \tilde{\mathscr{S}}|_{\partial\tilde{\mathscr{B}}}$ via the natural isomorphism $\det(\tilde{\mathscr{S}})|_{\partial\tilde{\mathscr{B}}} \cong \mathbb{R}v \otimes \det(\mathscr{S})$ for an outward pointing normal vector field v along $\partial\tilde{\mathscr{B}}$.

Example 2.6. Two regular G-moduli problems $(\mathscr{B}, \mathscr{E}_i, \mathscr{S}_i)$, $i = 0, 1$ (over the same base), are called *homotopic* if there exist a G-equivariant Hilbert space bundle $\mathscr{E} \to [0,1] \times \mathscr{B}$ and a G-equivariant smooth section $\mathscr{S} : [0,1] \times \mathscr{B} \to \mathscr{E}$ such that $\mathscr{E}_i = \mathscr{E}|_{\{i\} \times \mathscr{B}}$ and $\mathscr{S}_i = \mathscr{S}|_{\{i\} \times \mathscr{B}}$ for $i = 0, 1$, the triple $(\mathscr{B}, \mathscr{E}_t, \mathscr{S}_t)$, defined by $\mathscr{E}_t := \mathscr{E}|_{\{t\} \times \mathscr{B}}$ and $\mathscr{S}_t = \mathscr{S}|_{\{t\} \times \mathscr{B}}$, is a regular G-moduli problem for every $t \in [0,1]$, and the set $\mathscr{M} := \{(t,x) \in [0,1] \times \mathscr{B} \mid \mathscr{S}_t(x) = 0\}$ is compact. Note that two homotopic G-moduli problems are cobordant.

The next theorem is the main result of this paper. It states the properties of the Euler class. We denote by $H^*_{\mathrm{G}}(\mathscr{B})$ the equivariant cohomology (see Section 3) with real coefficients.

Theorem 2.7. *There exists a functor, called the* Euler class, *which assigns to each compact oriented Lie group* G *and each regular* G*-moduli problem* $(\mathscr{B},\mathscr{E},\mathscr{S})$ *a homomorphism* $\chi^{\mathscr{B},\mathscr{E},\mathscr{S}}: H^*_{\mathrm{G}}(\mathscr{B}) \to \mathbb{R}$ *and satisfies the following*:

(**Functoriality**) *If* (ψ,Ψ) *is a morphism from* $(\mathscr{B},\mathscr{E},\mathscr{S})$ *to* $(\mathscr{B}',\mathscr{E}',\mathscr{S}')$ *then* $\chi^{\mathscr{B},\mathscr{E},\mathscr{S}}(\psi^*\alpha) = \chi^{\mathscr{B}',\mathscr{E}',\mathscr{S}'}(\alpha)$ *for every* $\alpha \in H^*_{\mathrm{G}}(\mathscr{B}')$.

(**Thom class**) *If* (B,E,S) *is a finite dimensional oriented regular* G*-moduli problem and* $\tau \in \Omega^*_{\mathrm{G}}(E)$ *is an equivariant Thom form supported in an open neighbourhood* $U \subset E$ *of the zero section such that* $U \cap E_x$ *is convex for every* $x \in B$, $U \cap \pi^{-1}(K)$ *has compact closure for every compact set* $K \subset B$, *and* $S^{-1}(U)$ *has compact closure, then*

$$\chi^{B,E,S}(\alpha) = \int_{B/\mathrm{G}} \alpha \wedge S^*\tau$$

for every $\alpha \in H^*_{\mathrm{G}}(B)$.

(**Transversality**) *If* $\mathscr{S}$ *is transverse to the zero section, then*

$$\chi^{\mathscr{B},\mathscr{E},\mathscr{S}}(\alpha) = \int_{\mathscr{M}/\mathrm{G}} \alpha$$

for every $\alpha \in H^*_{\mathrm{G}}(B)$, *where* $\mathscr{M} := \mathscr{S}^{-1}(0)$.

(**Cobordism**) *If* $(\mathscr{B}_0,\mathscr{E}_0,\mathscr{S}_0)$ *and* $(\mathscr{B}_1,\mathscr{E}_1,\mathscr{S}_1)$ *are cobordant* G*-moduli problems, then*

$$\chi^{\mathscr{B}_0,\mathscr{E}_0,\mathscr{S}_0}(\iota_0^*\alpha) = \chi^{\mathscr{B}_0,\mathscr{E}_1,\mathscr{S}_1}(\iota_1^*\alpha)$$

for every $\alpha \in H^*_{\mathrm{G}}(\tilde{\mathscr{B}})$, *where* $\iota_0 : \mathscr{B}_0 \hookrightarrow \tilde{\mathscr{B}}$ *and* $\iota_1 : \mathscr{B}_1 \hookrightarrow \tilde{\mathscr{B}}$ *are the inclusions.*

(**Subgroup**) *If* $(\mathscr{B},\mathscr{E},\mathscr{S})$ *is a regular* G*-moduli problem and* $\mathrm{H} \subset \mathrm{G}$ *is a normal subgroup acting freely on* $\mathscr{B}$, *then*

$$\chi^{\mathscr{B}/\mathrm{H},\mathscr{E}/\mathrm{H},\mathscr{S}/\mathrm{H}}(\alpha) = \chi^{\mathscr{B},\mathscr{E},\mathscr{S}}(\alpha)$$

for every $\alpha \in H^*_{\mathrm{G/H}}(\mathscr{B}/\mathrm{H}) \cong H^*_{\mathrm{G}}(\mathscr{B})$.

(**Rationality**) *If* $\alpha \in H^*_{\mathrm{G}}(\mathscr{B};\mathbb{Q})$ *then* $\chi^{\mathscr{B},\mathscr{E},\mathscr{S}}(\alpha) \in \mathbb{Q}$.

The Euler class is uniquely determined by the (*Functoriality*) *and* (*Thom class*) *axioms.*

The integrals in the (*Transversality*) and (*Thom class*) axioms will be explained in Section 4 and the Thom class in Section 5.

3. Equivariant cohomology

3.1. Equivariant differential forms

Let B be a manifold and G be a compact Lie group acting smoothly on B. The (covariant) action of $g \in \mathrm{G}$ on B will be denoted by $\phi_g \in \mathrm{Diff}(B)$. We also use the notation $g^*x := \phi_{g^{-1}}(x)$ for the contravariant action. Let $\Omega^*_{\mathrm{G}}(B)$ denote the space of G-equivariant polynomials from $\mathfrak{g}$ to $\Omega^*(B)$. Thus the elements of $\Omega^*_{\mathrm{G}}(B)$ are maps $\alpha : \mathfrak{g} \to \Omega^*(B)$ that satisfy

$$\alpha(g^{-1}\xi g) = \phi_g^* \alpha(\xi)$$

for $\xi \in \mathfrak{g}$ and $g \in \mathrm{G}$. They are called *equivariant differential forms* on B. If $e_1, \ldots, e_n$ is a basis of $\mathfrak{g}$ and $\xi = \sum_{i=1}^n \xi^i e_i$, then $\alpha \in \Omega^\ell_{\mathrm{G}}(B)$ can be written in the form

$$\alpha(\xi) = \sum_I \xi^I \alpha_I,$$

where $I = (i_1, \ldots, i_n)$, $\xi^I = (\xi^1)^{i_1} \cdots (\xi^n)^{i_n}$, and $\alpha_I \in \Omega^{\ell - 2|I|}(B)$. The equivariant differential $d_{\mathrm{G}} : \Omega^\ell_{\mathrm{G}}(B) \to \Omega^{\ell+1}_{\mathrm{G}}(B)$ is defined by

$$(d_{\mathrm{G}}\alpha)(\xi) := d(\alpha(\xi)) + \iota(X_\xi)\alpha(\xi) = \sum_I \xi^I (d\alpha_I + \iota(X_\xi)\alpha_I)$$

for $\xi \in \mathfrak{g}$, where $X_\xi \in \mathrm{Vect}(B)$ denotes the covariant infinitesimal action, i.e. $X_\xi(x) := -\xi^* x$. The cohomology of this differential will be denoted by $H^*_{\mathrm{G}}(B)$. It is isomorphic to the singular cohomology of the space $B \times_{\mathrm{G}} \mathrm{EG}$ with real coefficients, where EG is a contractible space on which G acts freely and covariantly, and the action on $B \times \mathrm{EG}$ is given by $g^*(x, \theta) = (g^* x, g^{-1}\theta)$ for $x \in B$ and $\theta \in \mathrm{EG}$ (see [13]).

Standing hypothesis: In the remainder of this section $\mathrm{H} \subset \mathrm{G}$ is a normal subgroup which acts on B with finite isotropy.

We now introduce the notion of an H-basic equivariant differential form on B. If H acts freely on B then the H-basic forms are in one-to-one correspondence to the G/H-equivariant differential forms on B/H.

Definition 3.1. A form $\alpha \in \Omega^*_{\mathrm{G}}(B)$ is called H-*basic* if

$$\alpha(\xi + \eta) = \alpha(\xi), \qquad \iota(X_\eta)\alpha(\xi) = 0 \tag{2}$$

for all $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{h}$.

We need a simple lemma about Lie groups.

Lemma 3.2. *Let G be a Lie group and* $\mathrm{H} \subset \mathrm{G}$ *be a compact normal Lie subgroup. Then there exists an* H-*invariant complement of* $\mathfrak{h} = \mathrm{Lie}(\mathrm{H})$ *in* $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$. *Moreover,* H *acts trivially on every such complement. In particular,* $h^{-1}\xi h - \xi \in \mathfrak{h}$ *for all* $h \in \mathrm{H}$ *and* $\xi \in \mathfrak{g}$.

Proof. The existence of an H-invariant complement follows by averaging any projection $\pi : \mathfrak{g} \to \mathfrak{h}$ over H. How suppose that $\mathfrak{k}$ is such a complement. Let $\xi \in \mathfrak{k}$ and $h \in \mathrm{H}$ and suppose, by contradiction, that $h\xi h^{-1} \neq \xi$. Since $h\xi h^{-1} - \xi \in \mathfrak{k}$ it follows that $h\xi h^{-1} - \xi \notin \mathfrak{h}$. Hence there exists an $\varepsilon > 0$ such that $\exp(th\xi h^{-1})\exp(-t\xi) \notin \mathrm{H}$ for $0 < t \leqslant \varepsilon$. Hence $\exp(-t\xi)h\exp(t\xi) \notin \mathrm{H}$ for small positive t, a contradiction. □

Corollary 3.3. *Let* $\alpha \in \Omega^*_{\mathrm{G}}(B)$ *be* H-*basic. Then*

$$\alpha(\xi) = \phi_h^* \alpha(\xi) \tag{3}$$

for all $\xi \in \mathfrak{g}$ *and* $h \in \mathrm{H}$.

Proof. By Lemma 3.2, $h^{-1}\xi h - \xi \in \mathfrak{h}$ for every $h \in \mathrm{H}$ and $\xi \in \mathfrak{g}$. Hence $\phi_h^*\alpha(\xi) = \alpha(h^{-1}\xi h) = \alpha(\xi + h^{-1}\xi h - \xi) = \alpha(\xi)$ for $h \in \mathrm{H}$ and $\xi \in \mathfrak{g}$. □

We show below that the cohomology of the subcomplex of H-basic forms with differential d_{G} is isomorphic to the G-equivariant cohomology of B. This requires some preparation. Let $A \in \Omega^1(B, \mathfrak{h})$ be a G-equivariant H-connection. This means that

$$A_{g^*x}(g^*v) = g^{-1}A_x(v)g, \qquad A_x(\eta^*x) = \eta \tag{4}$$

for all $x \in B$, $v \in T_xB$, $g \in \mathrm{G}$, and $\eta \in \mathfrak{h}$.

Remark 3.4. By Lemma 3.2, every G-equivariant H-connection satisfies

$$h^{-1}\xi h - \xi = A_x((h^{-1}\xi h)^*x) - A_x(\xi^*x)$$

for $x \in B$, $\xi \in \mathfrak{g}$, and $h \in \mathrm{H}$.

Note that the covariant derivative d_A on $\Omega^*(B, \mathfrak{h})$ extends to $\Omega^*(B, \mathfrak{g})$ by the usual formula

$$d_A\Phi := d\Phi + [A \wedge \Phi]$$

for $\Phi \in \Omega^*(B, \mathfrak{g})$. The covariant derivative satisfies

$$d_A d_A \Phi = [F_A \wedge \Phi],$$

where $F_A \in \Omega^2(B, \mathfrak{h})$ is the curvature:

$$F_A := dA + \tfrac{1}{2}[A \wedge A].$$

Consider the space $\Omega^*_{\mathrm{G}}(B, \mathfrak{g})$ of G-equivariant polynomials $\Phi : \mathfrak{g} \to \Omega^*(B, \mathfrak{g})$. The equivariance condition means that

$$\Phi(g^{-1}\xi g) = g^{-1}(\phi_g^*\Phi(\xi))g \tag{5}$$

for $\xi \in \mathfrak{g}$ and $g \in \mathrm{G}$. It is interesting to consider the subspace of H-basic equivariant Lie algebra valued forms.

Definition 3.5. A form $\Phi \in \Omega^*_{\mathrm{G}}(B, \mathfrak{g})$ is called H-*basic* if

$$\Phi(\xi + \eta) = \Phi(\xi), \qquad \iota(X_\eta)\Phi(\xi) = 0 \tag{6}$$

for all $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{h}$.

Remark 3.6. By Lemma 3.2, every H-basic form $\Phi \in \Omega^*_{\mathrm{G}}(B, \mathfrak{g})$ satisfies

$$\Phi(\xi) = \Phi(h^{-1}\xi h) = h^{-1}(\phi_h^*\Phi(\xi))h$$

for $\xi \in \mathfrak{g}$ and $h \in \mathrm{H}$.

The subspace of H-basic forms is invariant under the operation $(\Phi, \Psi) \mapsto [\Phi \wedge \Psi]$. A G-equivariant H-connection A determines a covariant differential $d_{A,\mathrm{G}} : \Omega^*_{\mathrm{G}}(B, \mathfrak{g}) \to \Omega^{*+1}_{\mathrm{G}}(B, \mathfrak{g})$ defined by

$$(d_{A,\mathrm{G}}\Phi)(\xi) := d_A\Phi(\xi) + \iota(X_\xi)\Phi(\xi).$$

The *equivariant curvature* of A is defined as the 2-form $F_{A,\mathrm{G}} \in \Omega^2_{\mathrm{G}}(B, \mathfrak{g})$ given by

$$F_{A,\mathrm{G}}(\xi) := F_A + \xi + A(X_\xi).$$

Lemma 3.7. (i) *If Φ is* H*-basic then so is $d_{A,\mathrm{G}}\Phi$.*
(ii) *The curvature $F_{A,\mathrm{G}}$ is* H*-basic.*
(iii) *Every $\Phi \in \Omega^*_{\mathrm{G}}(B, \mathfrak{g})$ satisfies*

$$d_{A,\mathrm{G}}d_{A,\mathrm{G}}\Phi = [F_{A,\mathrm{G}} \wedge \Phi].$$

(iv) *The curvature satisfies the equivariant Bianchi identity*

$$d_{A,\mathrm{G}}F_{A,\mathrm{G}} = 0.$$

Proof. The first two assertion are obvious consequences of the definitions. Assertion (iii) follows from a computation:

$$\begin{aligned} d_{A,\mathrm{G}}d_{A,\mathrm{G}}\Phi(\xi) &= d_A d_A \Phi(\xi) + \iota(X_\xi)d_A\Phi(\xi) + d_A\iota(X_\xi)\Phi(\xi) \\ &= [F_A \wedge \Phi(\xi)] + \mathscr{L}_{X_\xi}\Phi(\xi) + \iota(X_\xi)[A \wedge \Phi(\xi)] + [A \wedge \iota(X_\xi)\Phi(\xi)] \\ &= [F_A \wedge \Phi(\xi)] + [\xi, \Phi(\xi)] + [A(X_\xi), \Phi(\xi)] \\ &= [F_{A,\mathrm{G}}(\xi) \wedge \Phi(\xi)]. \end{aligned}$$

In the third equality we have used the identity $\mathscr{L}_{X_\xi}\Phi(\xi) = [\xi, \Phi(\xi)]$ which follows from the G-equivariance of Φ.

We prove the Bianchi identity:

$$\begin{aligned} d_{A,\mathrm{G}}F_{A,\mathrm{G}}(\xi) &= d_A(F_A + \xi + A(X_\xi)) + \iota(X_\xi)F_A \\ &= d\iota(X_\xi)A + [A, \xi + A(X_\xi)] + \iota(X_\xi)dA + [A(X_\xi), A] \\ &= \mathscr{L}_{X_\xi}A + [A, \xi] \\ &= 0. \end{aligned}$$

Here the last equation follows from the G-equivariance of A. □

Now consider the operator $\Omega^*_{\mathrm{G}}(B) \to \Omega^*_{\mathrm{G}}(B) : \alpha \mapsto \alpha_A$ given by

$$\alpha_A(\xi) := (\pi_A^*\alpha)(F_{A,\mathrm{G}}(\xi)), \tag{7}$$

where $\pi_A : TB \to TB$ denotes the projection onto the kernel of A. Thus

$$\pi_{A,x}(v) := v - A_x(v)^*x$$

for $v \in T_xB$. More precisely, choose a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$, write $\alpha(\xi) = \sum_I \alpha_I \xi^I$, and denote

$$F^i(\xi) := F^i + \xi^i + A^i(X_\xi), \qquad A =: \sum_i A^i e_i, \qquad F_A =: \sum_i F^i e_i, \tag{8}$$

so that $F_{A,\mathrm{G}}(\xi) = \sum_i F^i(\xi) e_i$. Then α_A is given by

$$\alpha_A(\xi) = \sum_I F^I(\xi) \pi_A^* \alpha_I,$$

where $F^I(\xi) := F^1(\xi)^{i_1} \wedge \cdots \wedge F^n(\xi)^{i_n}$.

Theorem 3.8. *Let $A \in \Omega^1(B, \mathfrak{h})$ be a* G*-equivariant* H*-connection.*

(i) *If $\alpha \in \Omega^*_{\mathrm{G}}(B)$ then $\alpha_A : \mathfrak{g} \to \Omega^*(B)$ is* G*-equivariant and* H*-basic.*

(ii) *The operator $\alpha \mapsto \alpha_A$ is a d_{G}-chain map, i.e.*

$$d_{\mathrm{G}} \alpha_A = (d_{\mathrm{G}} \alpha)_A$$

for every $\alpha \in \Omega^*_{\mathrm{G}}(B)$.

(iii) *If $d_{\mathrm{G}} \alpha = 0$ and A' is another* G*-equivariant* H*-connection then there exists an* H*-basic form $\beta \in \Omega^*_{\mathrm{G}}(B)$ such that $\alpha_{A'} - \alpha_A = d_{\mathrm{G}} \beta$.*

(iv) *The operator $\alpha \mapsto \alpha_A$ is chain homotopic to the identity, i.e. there exists an operator $Q : \Omega^*_{\mathrm{G}}(B) \to \Omega^{*-1}_{\mathrm{G}}(B)$ such that*

$$\alpha - \alpha_A = d_{\mathrm{G}} Q \alpha + Q d_{\mathrm{G}} \alpha$$

for every $\alpha \in \Omega^*_{\mathrm{G}}(B)$.

Remark 3.9. If H acts freely on B then the H-basic forms are in one-to-one correspondence with G/H-equivariant differential forms on the quotient B/H. In this case the map $\alpha \mapsto \alpha_A$ induces an isomorphism from the G-equivariant cohomology of B to the G/H-equivariant cohomology of the quotient $B/\mathrm{H} : H^*_{\mathrm{G}}(B; \mathbb{R}) \cong H^*_{\mathrm{G/H}}(B/\mathrm{H}; \mathbb{R})$.

Remark 3.10. If $\mathrm{G} = \mathrm{H}$ acts with finite isotropy then the H-basic forms can be interpreted as differential forms on the quotient B/G which is now an orbifold. In the present paper we circumvent orbifold theory by always working on the total space B.

Remark 3.11. If $\ell > \dim B - \dim \mathrm{H}$ then every H-basic ℓ-form on B vanishes. Hence α_A is d_{G}-closed whenever $\deg(\alpha) = \dim B - \dim \mathrm{H}$.

Example 3.12. Assume $\mathrm{G} = \mathrm{H} = S^1$. Then the linear function $\alpha : \mathrm{i}\mathbb{R} \to \mathbb{R} \subset \Omega^0(B)$, given by

$$\alpha(\eta) := \frac{\mathrm{i}\eta}{2\pi},$$

is an S^1-closed equivariant 2-form on B. We claim that under the isomorphism

$$H^*_{S^1}(B) \cong H^*(B \times_{S^1} \mathrm{E}S^1)$$

the cohomology class of α corresponds to the pullback of the positive integral generator $c \in H^2(\mathrm{B}S^1; \mathbb{Z}) \cong \mathbb{Z}$ under the projection $\pi : B \times_{S^1} \mathrm{E}S^1 \to \mathrm{B}S^1$:

$$[\alpha] = \pi^* c.$$

To see this, note that $\pi^* c$ is the first Chern class of the line bundle $L := (B \times \mathrm{E}S^1 \times \mathbb{C})/S^1 \to B \times_{S^1} \mathrm{E}S^1$, where S^1 acts by $\lambda^*(x, \theta, \zeta) = (\lambda^* x, \lambda^{-1}\theta, \lambda^{-1}\zeta)$ for $x \in B$, $\theta \in \mathrm{E}S^1$, and $\zeta \in \mathbb{C}$. Now let $A \in \Omega^1(B, \mathrm{i}\mathbb{R})$ be a connection 1-form. Then

$$\alpha_A = \frac{\mathrm{i}F_A}{2\pi}.$$

This form descends to a 2-form on $B \times_{S^1} \mathrm{E}S^1$ which represents the first Chern class of L.

Proof of Theorem 3.8. Our proof is an adaptation of the argument in Section 5.1 of [13]. Let $e_1, \ldots, e_m$ be a basis of $\mathfrak{h}$ and denote by $X_i \in \mathrm{Vect}(B)$ the vector field $X_i(x) := -e_i^* x$. Consider the following operators on $\Omega^*_{\mathrm{G}}(B)$:

$$K\alpha(\xi) := -\sum_{i=1}^m A^i \wedge \partial_i \alpha(\xi),$$

$$R\alpha(\xi) := \sum_{i=1}^m dA^i \wedge \partial_i \alpha(\xi),$$

$$E_0\alpha(\xi) := -\sum_{i=1}^m A^i(X_\xi)\partial_i \alpha(\xi),$$

$$E_1\alpha(\xi) := -\sum_{i=1}^m A^i \wedge \iota(X_i)\alpha(\xi),$$

$$E := E_0 + E_1.$$

Note that the space of G-equivariant forms is preserved by all five operators. In the case of the operators K, R, and E_0 the proof relies on the identity

$$\phi_g^* \partial_i \alpha(\xi) = \sum_{j=1}^n (g^{-1} e_i g)^j \partial_j \alpha(g^{-1}\xi g),$$

where $n = \dim \mathfrak{g}$, $e_1, \ldots, e_n$ is an extension of the basis of $\mathfrak{h}$ to a basis of $\mathfrak{g}$, and ξ^i denotes the ith coordinate of $\xi \in \mathfrak{g}$ with respect to this basis. Note that with this notation $A^j = 0$ for $j > m$. As an example we prove equivariance in the case of E_0:

$$\begin{aligned}
-\phi_g^* E_0 \alpha(\xi) &= \sum_{i=1}^m \phi_g^* A^i(X_\xi) \phi_g^* \partial_i \alpha(\xi) \\
&= \sum_{i=1}^m \sum_{j=1}^n \phi_g^* A^i(X_\xi)(g^{-1} e_i g)^j \partial_j \alpha(g^{-1}\xi g) \\
&= \sum_{j=1}^n \left(\sum_{i=1}^m \phi_g^* A^i(X_\xi)(g^{-1} e_i g)^j \right) \partial_j \alpha(g^{-1}\xi g)
\end{aligned}$$

$$= \sum_{j=1}^{n} \phi_g^*(g^{-1}A(X_\xi)g)^j \partial_j \alpha(g^{-1}\xi g)$$

$$= \sum_{j=1}^{n} A^j(X_{g^{-1}\xi g})\partial_j \alpha(g^{-1}\xi g)$$

$$= -E_0\alpha(g^{-1}\xi g).$$

The operators K, R, and E satisfy the following crucial identity

$$d_{\mathrm{G}}K + Kd_{\mathrm{G}} = E - R. \tag{9}$$

The proof is a straightforward computation. We shall prove that the kernel of E is the space of H-basic forms:

$$E\alpha = 0 \quad \Leftrightarrow \quad \alpha \text{ is H-basic.} \tag{10}$$

To see this we observe that the operators E_0 and E_1 commute and that $\Omega^*_{\mathrm{G}}(B)$ decomposes as a direct sum

$$\Omega^*_{\mathrm{G}}(B) = \bigoplus_{p,q} \Omega^{p,q},$$

where $E_0\alpha = p\alpha$ and $E_1\alpha = q\alpha$ for every $\alpha \in \Omega^{p,q}$. To describe the space $\Omega^{p,q}$ we choose frames $e_{m+1}(x), \ldots, e_n(x)$ in $\mathfrak{g}$ depending smoothly on $x \in B$ such that

$$A_x(e_j(x)^*x) = 0$$

for every $j > m$. It follows that the vectors $e_1, \ldots, e_m, e_{m+1}(x), \ldots, e_n(x)$ form a basis of $\mathfrak{g}$ for every x. In this basis $\Omega^{p,q}$ is generated by monomials of the form

$$A^{k_1} \wedge \cdots \wedge A^{k_q} \wedge \alpha\eta^I\zeta^J,$$

where $|I| = p$ and $\alpha \in \Omega^*(B)$ is H-horizontal. Here we use the notation

$$\xi = \sum_{i \leqslant m} \eta^i e_i + \sum_{j > m} \zeta^j e_j(x). \tag{11}$$

It follows that the kernel of E is $\Omega^{0,0}$ and this proves (10). We denote by

$$\pi : \Omega^*_{\mathrm{G}}(B) \to \Omega^{0,0}$$

the projection onto the kernel of E along the direct sum of the spaces $\Omega^{p,q}$ for $p + q > 0$. An explicit formula for π with respect to the above frame is

$$\pi\left(\sum_{I,J} \alpha_{I,J}\eta^I\zeta^J\right) = \sum_J \pi_A^*\alpha_{\emptyset,J}\zeta^J.$$

This discussion shows that the operator $\pi + E$ is invertible and preserves the (p,q)-degree.

From now on the argument is exactly the same as in [13]. We reproduce it here since it is short and beautiful. Since R lowers the p-degree it follows that the operator $(\pi + E)^{-1}R$ is nilpotent and hence $\pi + E - R$ is invertible. Denote

$$U := (\pi + E - R)^{-1}, \qquad Q := KU.$$

Then we obtain

$$[d_{\mathrm{G}}, U] = [\pi, d_{\mathrm{G}}]U. \tag{12}$$

Here we use the fact that, by (9), the operator $E-R$ commutes with d_{G}, hence $[\pi+E-R, d_{\mathrm{G}}]=[\pi, d_{\mathrm{G}}]$, and hence $[d_{\mathrm{G}}, U] = U[\pi, d_{\mathrm{G}}]U$. Now Eq. (12) follows from the fact that U acts as the identity on $\Omega^{0,0}$ and the image of $[\pi, d_{\mathrm{G}}]$ is contained in $\Omega^{0,0}$. Moreover, it is obvious from the definitions that K vanishes on $\Omega^{0,0}$ and so $K[\pi, d_{\mathrm{G}}] = 0$. Hence

$$\begin{aligned}
d_{\mathrm{G}}Q + Qd_{\mathrm{G}} &= d_{\mathrm{G}}KU + KUd_{\mathrm{G}} \\
&= d_{\mathrm{G}}KU + KUd_{\mathrm{G}} + K[\pi, d_{\mathrm{G}}]U \\
&= d_{\mathrm{G}}KU + KUd_{\mathrm{G}} + K[d_{\mathrm{G}}, U] \\
&= (d_{\mathrm{G}}K + Kd_{\mathrm{G}})U \\
&= (E - R)U \\
&= \mathrm{id} - \pi U.
\end{aligned}$$

To complete the proof of (iv) we must show that

$$\pi U\alpha = \alpha_A \tag{13}$$

for every $\alpha \in \Omega^*_{\mathrm{G}}(B)$. It suffices to prove (13) for a monomial

$$\alpha = \alpha_{I,J}\eta^I\zeta^J.$$

Write πU in the form

$$\pi U = \pi(\pi + E)^{-1}(\mathrm{id} + R(\pi + E)^{-1} + (R(\pi + E)^{-1})^2 + \cdots).$$

Since $R(\pi + E)^{-1}$ lowers the p-degree by one and $\pi(\pi + E)^{-1} = \pi$, it follows that

$$\pi U\alpha = \pi(R(\pi + E)^{-1})^{\ell}\alpha,$$

where $\ell = |I|$. Now consider the operator given by

$$S := \sum_i F^i \wedge \partial_i.$$

Then

$$S - R = \frac{1}{2}\sum_{i,j,k} c_{ij}^k A^i \wedge A^j \wedge \partial_k,$$

where c_{ij}^k are the structure constants of $\mathfrak{g}$ defined by $[e_i, e_j]=\sum_k c_{ij}^k e_k$. Since $S-R$ raises the q-degree by two, we have

$$\begin{aligned}
\pi U\alpha &= \pi(S(\pi + E)^{-1})^{\ell}\alpha \\
&= (S(\pi + E)^{-1})^{\ell}\pi_A^*\alpha \\
&= \frac{1}{\ell!}S^{\ell}\pi_A^*\alpha_{I,J}\eta^I\zeta^J \\
&= \pi_A^*\alpha_{I,J} \wedge F^I\zeta^J.
\end{aligned}$$

To see that this is the required formula we write ξ in the form (11) and note that, since $A_x(e_j(x)^*x)=0$ for $j>m$, we have

$$\xi + A(X_\xi) = \sum_{j=m+1}^{n} \zeta^j e_j(x).$$

Hence

$$F^i(\xi) = F^i + \xi^i + A^i(X_\xi) = \begin{cases} F^i & \text{for } i \leqslant m, \\ \zeta^i & \text{for } i > m. \end{cases}$$

This proves (iv). Assertion (ii) is an obvious consequence of (iv). Assertion (i) follows from the fact that operators π, E, and R preserve the space of G-equivariant forms.

We prove (iii). Let $t \mapsto A_t$ be a smooth family of G-equivariant H-connections. Think of the path $t \mapsto A_t$ as a connection $\tilde{A}$ on the space $\tilde{B}:=\mathbb{R} \times B$. Given a G-closed ℓ-form $\alpha \in \Omega_{\mathrm{G}}^*(B)$ denote

$$\tilde{\alpha}(\xi):=\alpha_{\tilde{A}}(\xi)=:\alpha_t(\xi) + \mathrm{d}t \wedge \beta_t(\xi),$$

where $\alpha_t = \alpha_{A_t} \in \Omega_{\mathrm{G}}^{\ell}(B)$ and $\beta_t \in \Omega_{\mathrm{G}}^{\ell-1}(B)$. By assertion (ii), $\tilde{\alpha}$ is G-closed and, by assertion (i), it is G-invariant and H-basic. Hence α_t and β_t are G-invariant and H-basic, α_t is G-closed, and $\partial_t\alpha_t = d_{\mathrm{G}}\beta_t$ for every t. Hence

$$\alpha_{A_1} - \alpha_{A_0} = d_{\mathrm{G}} \int_0^1 \beta_t \, \mathrm{d}t.$$

Since β_t is H-basic for every t, this proves (iii). $\square$

4. Invariant integration

Throughout this section we assume that B is a finite dimensional oriented manifold, that G is a compact oriented Lie group acting on B by orientation preserving diffeomorphisms, and that the isotropy subgroups are finite. Integration requires the notion of local slices whose existence the next theorem asserts. A proof can be found in [3].

Theorem 4.1. *Suppose* G *acts on the finite dimensional manifold B with finite isotropy and let* $m:=\dim B - \dim \mathrm{G}$. *Then, for every $x_0 \in B$, there exists a triple $(U_0, \phi_0, \mathrm{G}_0)$ with the following properties*:

(i) $\mathrm{G}_0 \subset \mathrm{G}$ *is a finite subgroup.*
(ii) $U_0 \subset H_0$ *is a* G_0*-invariant open neighbourhood of zero in an oriented m-dimensional real Hilbert space H_0 with an orthogonal linear action of* G_0.
(iii) $\phi_0 : U_0 \to B$ *is a* G_0*-equivariant embedding such that $x_0 = \phi_0(0)$ and the induced map* $\mathrm{G} \times_{\mathrm{G}_0} U_0 \to B : [g,x] \mapsto g^*\phi_0(x)$ *is an orientation preserving diffeomorphism onto a* G*-invariant open neighbourhood of x_0. Here the equivalence relation is* $[g,x] = [g_0^{-1}g, g_0^*x]$.

A triple $(U_0, \phi_0, \mathrm{G}_0)$ with these properties is called a local slice.

We now explain how to integrate invariant and horizontal m-forms on B over the quotient B/G. Suppose that $\alpha \in \Omega^m_{\mathrm{G}}(B)$ is an equivariant m-form with compact support. Choose finitely many local slices $(U_i, \phi_i, \mathrm{G}_i)$, $i = 1, \dots, N$, such that the open sets $\mathrm{G}^*\phi_i(U_i)$ cover the support of α, and define

$$\int_{B/\mathrm{G}} \alpha := \sum_{i=1}^{N} \frac{1}{|\mathrm{G}_i|} \int_{U_i} \phi_i^*(\rho_i \alpha_A), \tag{14}$$

where $A \in \Omega^1(B, \mathfrak{g})$ is a G-connection, α_A is defined by (7), and the functions $\rho_i : B \to [0,1]$ are G-invariant and form a partition of unity such that $\operatorname{supp} \rho_i \subset \mathrm{G}^*\phi_i(U_i)$. The next proposition asserts that integral (14) is well defined and depends only on the (compactly supported) cohomology class of α.

Proposition 4.2. (i) *The right-hand side of* (14) *is independent of the local slices, the partition of unity, and the connection used to define it.*

(ii) *If B is a manifold with boundary and $\beta \in \Omega^{m-1}_{\mathrm{G}}(B)$ has compact support then*

$$\int_{B/\mathrm{G}} d_{\mathrm{G}}\beta = \int_{\partial B/\mathrm{G}} \beta.$$

Proof. We prove that the integral is independent of the choice of the local slices and the partition of unity. Let $(U_0, \phi_0, \mathrm{G}_0)$ and $(U_1, \phi_1, \mathrm{G}_1)$ be two local slices and suppose that α is supported in $\mathrm{G}^*\phi_0(U_0) \cap \mathrm{G}^*\phi_1(U_1)$. Shrinking U_0 and U_1, if necessary, we may assume that $\mathrm{G}^*\phi_0(U_0) = \mathrm{G}^*\phi_1(U_1)$. By definition, the map $U_0 \times \mathrm{G} \to B : (x_0, g) \mapsto g^*\phi_0(x_0)$ is an immersion and is transverse to ϕ_1. Hence the set

$$W := \{(x_0, x_1, g) \in U_0 \times U_1 \times \mathrm{G} \mid g^*\phi_0(x_0) = \phi_1(x_1)\}$$

is a smooth oriented m-manifold and

$$(x_0, x_1, g) \in W \quad \Rightarrow \quad (g_0^* x_0, x_1, g_0^{-1} g), (x_0, g_1^* x_1, g g_1) \in W$$

for $g_0 \in \mathrm{G}_0$ and $g_1 \in \mathrm{G}_1$. It follows that the projection $\pi_0 : W \to U_0$ is an orientation preserving submersion of degree $|\mathrm{G}_1|$ and the projection $\pi_1 : W \to U_1$ is an orientation preserving submersion of degree $|\mathrm{G}_0|$. Moreover, these projections satisfy

$$\phi_1 \circ \pi_1(x_0, x_1, g) = g^*(\phi_0 \circ \pi_0(x_0, x_1, g)).$$

This means that the maps $\phi_1 \circ \pi_1 : W \to B$ and $\phi_0 \circ \pi_0 : W \to B$ are related by the gauge transformation $W \to \mathrm{G} : (x_0, x_1, g) \mapsto g$. Since the form $\alpha_A \in \Omega^m(B)$ is invariant and horizontal this implies that

$$(\phi_0 \circ \pi_0)^* \alpha_A = (\phi_1 \circ \pi_1)^* \alpha_A \in \Omega^m(W).$$

Hence

$$|\mathrm{G}_0| \int_{U_1} \phi_1^* \alpha_A = \int_W \pi_1^* \phi_1^* \alpha_A = \int_W \pi_0^* \phi_0^* \alpha_A = |\mathrm{G}_1| \int_{U_0} \phi_0^* \alpha_A.$$

This proves that the right-hand side of (14) is independent of the local slices $(U_i, \phi_i, \mathrm{G}_i)$ and the partition of unity used to define it. Assertion (ii) follows from Stokes' theorem and Theorem 3.8(ii) whenever β is supported in the G-orbit of the image of a local slice. In general it follows by considering the sum $\sum_i d_G(\rho_i\beta)$ for a partition of unity ρ_i. That the right-hand side of (14) is independent of A follows from Theorem 3.8(iii). $\square$

Example 4.3. Consider the action of $G:=\mathbb{Z}_2$ on $B:=\mathbb{R}$ by $x \mapsto -x$. Then the identity map $\mathbb{R} \to B=\mathbb{R}$ is a local slice (or in fact a global slice). An equivariant differential form is a $\mathbb{Z}_2$-invariant differential form on $\mathbb{R}$. Consider the equivariant 1-form $\alpha = f(x)\,\mathrm{d}x$ where $f : \mathbb{R} \to \mathbb{R}$ has compact support and $f(x) = f(-x)$. Then

$$\int_{\mathbb{R}/\mathbb{Z}_2} \alpha = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

5. Thom forms

In [2] Atiyah and Bott noted that the Thom isomorphism theorem extends to equivariant cohomology and gives an isomorphism

$$H^i_G(E, E \setminus B) \to H^{i-n}_G(B).$$

Here H^*_G denotes equivariant cohomology with real coefficients, $E \to B$ is an oriented G-vector bundle and B is embedded into E as the zero section. In terms of the de Rham model the (equivariant) cohomology of the pair $(E, E \setminus B)$ is isomorphic to the (equivariant) de Rham cohomology of E with *vertical compact support*. In the nonequivariant case the isomorphism is established in [4, Theorem 6.17]. In [13, Chapter 10] Guillemin and Sternberg construct an equivariant Thom class and prove the Thom isomorphism theorem in the equivariant context. Below we give an alternative construction of the equivariant Thom class.

Definition 5.1. Let (B, E, S) be a finite dimensional oriented G-moduli problem and set $n := \operatorname{rank} E$. A *Thom structure* on (B, E, S) is a pair (U, τ) with the following properties:

(i) $U \subset E$ is a G-invariant open neighbourhood of the zero section that intersects each fibre in a convex set. Moreover, $U \cap E|_K$ has compact closure for every compact subset $K \subset B$.
(ii) $S^{-1}(U)$ has compact closure.
(iii) $\tau \in \Omega^n_G(E)$ is an equivariant n-form such that

$$d_G\tau = 0, \qquad \operatorname{supp}(\tau) \subset U, \qquad \int_{E_x} \tau = 1$$

for every $x \in B$.

Note that an equivariant n-form on E can be expressed as

$$\tau(\xi) = \sum_{k=0}^{[n/2]} \tau_k(\xi),$$

where $\tau_k : \mathfrak{g} \to \Omega^{n-2k}(E)$ is a homogeneous G-equivariant polynomial of degree k. The integral in (iii) is to be understood as the integral of the leading term $\tau_0 \in \Omega^n(E)$. We emphasize that in the case of nontrivial finite isotropy this integral does not agree with (14). It is a special case of integration over the fibre discussed in Section 6.

Remark 5.2. Suppose that (B, E, S) is a finite dimensional regular G-moduli problem. Let $A \in \Omega^1(E, \mathfrak{g})$ be a G-connection and (U, τ) be a Thom structure. Then $\tau_A \in \Omega^n(E)$ is a G-invariant and horizontal n-form. It is supported in U and, by Theorem 3.8, τ_A is closed. Moreover,

$$\int_{E_x} \tau_A = 1 \tag{15}$$

for every $x \in B$. To see this, recall that the isotropy subgroup G_x is finite. Thus the connection can be chosen such that the tangent vectors to E_x are horizontal. Then the curvature of A vanishes on E_x and so the restriction of τ_A to E_x agrees with the leading term τ_0. By Theorem 3.8(iii), the integral of τ_A over E_x is independent of the connection A and this proves (15).

Theorem 5.3. *Let (B, E, S) be a finite dimensional oriented* G*-moduli problem. Then (B, E, S) admits a Thom structure. Moreover, if (U_0, τ_0) and (U_1, τ_1) are two Thom structures then there exists an equivariant $(n-1)$-form $\sigma \in \Omega_{\mathrm{G}}^{n-1}(E)$ such that* $\operatorname{supp} \sigma \subset U_0 \cup U_1$ *and $d_{\mathrm{G}}\sigma = \tau_1 - \tau_0$.*

The construction of a Thom structure is based on the existence of an $\mathrm{SO}(n)$-equivariant *universal Thom form* on $\mathbb{R}^n$. For completeness, we present an alternative proof to the one given in [13].

Proposition 5.4 (Guillemin and Sternberg [13]). *There exists a $d_{\mathrm{SO}(n)}$-closed form $\rho \in \Omega^n_{\mathrm{SO}(n)}(\mathbb{R}^n)$ with compact support and integral one (of the leading term). This form is called the* universal Thom form.

Proof. We look for ρ in the form

$$\rho(\eta) = \sum_k f_k(|x|^2/2)\rho_k(\eta),$$

where $\rho_k(\eta) \in \Omega^{n-2k}(\mathbb{R}^n)$ are forms with constant coefficients, and $f_k : [0, \infty) \to \mathbb{R}$ are smooth functions with compact support. Then

$$d_{\mathrm{SO}(n)}\rho(\eta) = \sum_k f_k'(|x|^2/2)\lambda \wedge \rho_k(\eta) + f_k(|x|^2/2)\iota(X_\eta)\rho_k(\eta),$$

where

$$\lambda := d(|x|^2/2) = \sum_{i=1}^n x_i \, \mathrm{d}x_i \in \Omega^1(\mathbb{R}^n).$$

So ρ will be $d_{\mathrm{SO}(n)}$-closed provided that

$$\iota(X_\eta)\rho_k(\eta) = \lambda \wedge \rho_{k+1}(\eta) \tag{16}$$

and

$$f_k'(s) + f_{k-1}(s) = 0.$$

The existence of forms $\rho(\eta)$ satisfying Eq. (16) is proved in Lemma 5.5. The functions f_k are constructed inductively. Choose a smooth function $f_0 : [0, \infty) \to [0, \infty)$ with compact support such that $f_0(r^2/2) = 0$ for $r < \delta$ and $r \geqslant 1$, and

$$\int_0^\infty f_0(r^2/2) \operatorname{Vol}(S^{n-1}) r^{n-1} \, \mathrm{d}r = 1.$$

Now define $f_k : [0,\infty) \to \mathbb{R}$ for $1 \leqslant k \leqslant n/2$ inductively by

$$f_k'(s) + f_{k-1}(s) = 0, \qquad f_k(1) = 0.$$

This implies

$$f_k(0) = \frac{1}{(k-1)!}\int_0^\infty s^{k-1} f_0(s)\,\mathrm{d}s = \frac{1}{2^{k-1}(k-1)!}\int_0^\infty r^{2k-1} f_0(r^2/2)\,\mathrm{d}r.$$

So for $k < n/2$ the functions $f_k(s)$ will vanish for $s < \delta$ provided that

$$\int_0^\infty s^{k-1} f_0(s)\,\mathrm{d}s = 0, \qquad 1 \leqslant k < n/2.$$

This can be achieved because the polynomials s^{k-1} are linearly independent. Note that, if n is odd, then f_k vanishes near zero for all k but, if n is even, then $f_{n/2}(0) = 1/2^{n/2-1}(n/2-1)!\mathrm{Vol}(S^{n-1}) > 0$. □

It remains to prove the lemma used in the preceding proof.

Lemma 5.5. *For $\eta = -\eta^T \in \mathbb{R}^{n\times n}$ and $k \in \mathbb{N}$ let $X_\eta \in \mathrm{Vect}(\mathbb{R}^n)$, $\omega_\eta \in \Omega^2(\mathbb{R}^n)$, and $\rho_k(\eta) \in \Omega^{n-2k}(\mathbb{R}^n)$ be given by*

$$X_\eta(x) := \eta x, \qquad \omega_\eta := \sum_{i<j} \eta_{ij}\,\mathrm{d}x_i \wedge \mathrm{d}x_j, \qquad \rho_k(\eta) := \frac{1}{k!} * \omega_\eta^k,$$

where $*$ denotes the Hodge $*$-operator with respect to the standard metric. Then the forms ρ_k satisfy (16), *i.e.*

$$\iota(X_\eta)\rho_k(\eta) = \lambda \wedge \rho_{k+1}(\eta),$$

where $\lambda := \sum_{i=1}^n x_i\,\mathrm{d}x_i \in \Omega^1(\mathbb{R}^n)$.

Proof. Since there is an obvious inclusion of $\mathrm{SO}(n)$ into $\mathrm{SO}(n+1)$, the statement for $n+1$ implies the statement for n. Thus it suffices to prove the lemma in the case where n is even. Since both sides of Eq. (16) are equivariant polynomials on $\mathfrak{so}(n)$ with values in $\Omega^{n-2k-1}(\mathbb{R}^n)$, it suffices to prove the lemma for elements of a maximal torus in $\mathfrak{so}(n)$. Assume $n = 2\ell$ and consider the maximal torus $T \subset \mathrm{SO}(2\ell)$ whose Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$ consists of matrices of the form

$$\eta = \mathrm{diag}(-\mathrm{i}\eta_1, \ldots, -\mathrm{i}\eta_\ell).$$

Here we identify $\mathbb{R}^{2\ell}$ with $\mathbb{C}^\ell$. Write the coordinates on $\mathbb{R}^{2\ell}$ in the form $(x_1, y_1, \ldots, x_\ell, y_\ell)$ and denote $\omega_i := \mathrm{d}x_i \wedge \mathrm{d}y_i$. Then, for $\eta \in \mathfrak{t}$,

$$\omega_\eta = \sum_i \eta_i \omega_i, \qquad \frac{1}{k!}\omega_\eta^k = \sum_{i_1 < \cdots < i_k} \eta_{i_1} \cdots \eta_{i_k}\,\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}.$$

Assume $\eta_i \neq 0$ for every i and denote $\tilde{\eta}_i := 1/\eta_i$. Then

$$\rho_k(\eta) = \frac{\eta_1 \cdots \eta_\ell}{(\ell - k)!}\,\omega_{\tilde{\eta}}^{\ell-k}, \qquad \iota(X_\eta)\omega_{\tilde{\eta}} = \lambda.$$

Hence, in this case

$$\begin{aligned}\iota(X_\eta)\rho_k(\eta) &= \frac{\eta_1\cdots\eta_\ell}{(\ell-k)!}\,\iota(X_\eta)\omega_{\tilde\eta}^{\ell-k}\\ &= \frac{\eta_1\cdots\eta_\ell}{(\ell-k-1)!}(\iota(X_\eta)\omega_{\tilde\eta})\wedge\omega_{\tilde\eta}^{\ell-k-1}\\ &= \lambda\wedge\rho_{k+1}(\eta).\end{aligned}$$

This proves the lemma for every $\eta\in\mathfrak{t}$ such that $\eta_i\neq 0$ for all i. For general elements $\eta\in\mathfrak{t}$ Eq. (16) follows by continuity. □

Proof of Theorem 5.3. Proof Let $\pi : P \to B$ be the bundle of oriented orthonormal frames of E. The fibre of P over $x\in B$ is the space

$$P_x := \{p : \mathbb{R}^n \to E_x \mid p \text{ preserves orientation and norm}\}.$$

Then P is a principal $\mathrm{SO}(n)$-bundle and E is isomorphic to $P\times_{\mathrm{SO}(n)}\mathbb{R}^n$. Since G acts on the fibres of E by orientation preserving isomorphisms there is an induced action of G on P. Thus $\mathrm{G}\times\mathrm{SO}(n)$ acts on $P\times\mathbb{R}^n$ by

$$(g,a)^*(x,p,v) := (g^*x, g^*pa, a^{-1}v).$$

Note that the actions of G and $\mathrm{SO}(n)$ commute, the action of $\mathrm{SO}(n)$ is free, and the projection $\pi : P \to B$ is G-equivariant. The universal Thom class ρ pulls back under the projection $P\times\mathbb{R}^n\to\mathbb{R}^n$ to a $(\mathrm{G}\times\mathrm{SO}(n))$-equivariant Thom form (still denoted by ρ) on $P\times\mathbb{R}^n$. Here the polynomial map $\rho : \mathfrak{g}\times\mathfrak{so}(n)\to\Omega^*(P\times\mathbb{R}^n)$ is independent of the $\mathfrak{g}$-variables.

Now let $A\in\Omega^1(P\times\mathbb{R}^n,\mathfrak{so}(n))$ be a $(\mathrm{G}\times\mathrm{SO}(n))$- equivariant $\mathrm{SO}(n)$-connection. Define $\rho_A\in\Omega^*_{\mathrm{G}\times\mathrm{SO}(n)}(P\times\mathbb{R}^n)$ by (7). Then, by Theorem 3.8(i), ρ_A is $\mathrm{SO}(n)$-basic and so descends to a G-equivariant differential form τ' on $P\times_{\mathrm{SO}(n)}\mathbb{R}^n\cong E$. By Theorem 3.8(ii), the form τ' is d_{G}-closed. Moreover, by construction, it has vertical compact support and integral one over each fibre. This proves the existence of a Thom form $\tau'\in\Omega^n_{\mathrm{G}}(E)$ with support in a neighbourhood $U'\subset E$ of the zero section that satisfies (i) but not necessarily (ii).

Let $U\subset E$ be an open neighbourhood of the zero section that satisfies (i) and (ii). We prove the existence of a Thom form τ with support in U. Choose a G-invariant function $f : B\to[0,\infty)$ such that $\mathrm{e}^{-f}U'\subset U$ and consider the G-equivariant isotopy $\psi_t : E\to E$ given by

$$\psi_t(x,v) := (x,\mathrm{e}^{tf(x)}v).$$

Then ψ_t is the flow of the G-invariant vector field $X\in\mathrm{Vect}(E)$ defined by $X(x,v):=(0,f(x)v)$ and

$$\tau := \psi_1^*\tau'$$

is a Thom form with support in U. Moreover,

$$\tau-\tau' = d_{\mathrm{G}}\sigma',\qquad \sigma' := \int_0^1 \psi_t^*\iota(X)\tau'\,\mathrm{d}t.$$

Thus σ' is an equivariant $(n-1)$-form on E with support in U'.

We prove that the difference of two Thom forms τ_0 and τ_1 is exact. To see this we assume, without loss of generality, that B is connected and use the equivariant version of the Thom isomorphism theorem [4, Theorem 6.17] as in [13, Chapter 10]. It asserts that there is an isomorphism

$$H^n_{\mathrm{G,vc}}(E) \cong H^n_{\mathrm{G}}(E, E \setminus B) \cong H^0_{\mathrm{G}}(B) \cong \mathbb{R}.$$

Here the subscript vc stands for *vertical compact support*. Since integration over the fibre defines a nontrivial homomorphism

$$H^n_{\mathrm{G,vc}}(E) \to \mathbb{R} : \tau \mapsto \int_{E_x} \tau$$

and the cohomology class $[\tau_1 - \tau_0]$ lies in the kernel of this homomorphism, it follows that $[\tau_1 - \tau_0] = 0 \in H^n_{\mathrm{G,vc}}(E)$. This means that there exists an equivariant $(n-1)$-form $\sigma \in \Omega^{n-1}_{\mathrm{G,vc}}(E)$ with vertical compact support such that $\tau_1 - \tau_0 = d_{\mathrm{G}}\sigma$. We prove that σ can be chosen with support in $U_0 \cup U_1$. To see this, choose a G-equivariant diffeomorphism $\psi = \psi_1$ as above. Then $\psi^*\tau_i - \tau_i = d_{\mathrm{G}}\sigma_i$ for $i = 0, 1$, where $\sigma_i \in \Omega^{n-1}_{\mathrm{G}}(E)$ is supported in U_i. Moreover the function $f : B \to [0, \infty)$ can be chosen so large that the form $\psi^*\sigma$ is supported in $U_0 \cup U_1$. Hence

$$\begin{aligned}\tau_1 - \tau_0 &= \tau_1 - \psi^*\tau_1 + \psi^*(\tau_1 - \tau_0) + \psi^*\tau_0 - \tau_0\\ &= d_{\mathrm{G}}(\sigma_0 + \psi^*\sigma - \sigma_1).\end{aligned}$$

This proves the theorem. □

Let (B, E, S) be a finite dimensional oriented regular G-moduli problem and (U, τ) be a Thom structure. We define a homomorphism

$$\chi^{B,E,S} : H^*_{\mathrm{G}}(B; \mathbb{R}) \to \mathbb{R}$$

by

$$\chi^{B,E,S}(\alpha) := \int_{B/\mathrm{G}} \alpha \wedge S^*\tau \tag{17}$$

for every equivariantly closed form $\alpha \in \Omega^*_{\mathrm{G}}(B)$. By Theorem 5.3 the number $\chi^{B,E,S}(\alpha)$ is independent of the Thom structure (U, τ) used to define it.

Example 5.6. Consider the trivial bundle $E := B \times \mathbb{R}$ over $B := \mathbb{R}$ and the section

$$S(x) := \arctan(x)$$

(so $S(\pm\infty) = \pm\pi/2$). Denote by y the variable in the fibre. An example of a Thom structure is

$$U := \mathbb{R} \times (-1, 1), \qquad \tau := \rho(y)\,\mathrm{d}y,$$

where $\rho : \mathbb{R} \to \mathbb{R}$ is an even function with integral one whose support is contained in the interval $(-1, 1)$. The map

$$\chi^{B,E,S} : H^0(\mathbb{R}) \cong \mathbb{R} \to \mathbb{R}$$

is multiplication by one. If the bundle E is equipped with the $\mathbb{Z}_2$-action $(x, y) \mapsto (-x, -y)$ then the invariant is multiplication by 1/2.

Now consider the neighbourhood

$$U' := \mathbb{R} \times ((-3,-2) \cup (-1,1) \cup (2,3))$$

and the differential form $\tau' := \rho'(y)\,\mathrm{d}y$ where $\rho' : \mathbb{R} \to \mathbb{R}$ is an even function with integral one and support in the union of the intervals $(-3,-2)$ and $(2,3)$. This pair (U', τ') violates the convexity hypothesis in Definition 5.1. The pullback form $S^*\tau'$ vanishes and so integrating it gives the wrong answer for $\chi^{B,E,S}$, namely zero.

6. Integration over the fibre

Throughout this section we assume that $\pi : E \to B$ is an oriented finite dimensional real vector bundle of rank n over a smooth oriented manifold, that G is a compact oriented Lie group acting on B and E by orientation preserving diffeomorphisms, and that π is equivariant. We denote by $\Omega^*_{\mathrm{G,vc}}(E)$ the space of equivariant differential forms on E with vertical compact support. This means that for every compact subset $K \subset B$ the support of the differential form intersects $\pi^{-1}(K)$ in a compact set.

The next theorem introduces *integration over the fibre* for equivariant differential forms. The corresponding map on the cohomology level exists in much greater generality [2].

Theorem 6.1. *There exists a linear map*

$$\pi_* : \Omega^*_{\mathrm{G,vc}}(E) \to \Omega^{*-n}_{\mathrm{G}}(B)$$

with the following properties.

(**Chain map**) $d_{\mathrm{G}} \circ \pi_* = \pi_* \circ d_{\mathrm{G}}$.

(**Thom class**) *If* $\tau \in \Omega^n_{\mathrm{G,vc}}(E)$ *is a Thom form then* $\pi_*\tau = 1$.

(**Module structure**) *For* $\alpha \in \Omega^*_{\mathrm{G,vc}}(E)$ *and* $\beta \in \Omega^*_{\mathrm{G}}(B)$,

$$\pi_*(\pi^*\beta \wedge \alpha) = \beta \wedge \pi_*\alpha.$$

(**Connection**) *If* G *acts on B with finite isotropy then, for every* $\alpha \in \Omega^*_{\mathrm{G,vc}}(E)$ *and every connection* 1-*form* $A \in \Omega^1(B, \mathfrak{g})$,

$$\pi_*\alpha_{\pi^*A} = (\pi_*\alpha)_A.$$

(**Functoriality**) *If* G *acts on B with finite isotropy and* $\alpha \in \Omega^{\dim B + n}_{\mathrm{G,vc}}(E)$ *has compact support then*

$$\int_{E/\mathrm{G}} \alpha = \int_{B/\mathrm{G}} \pi_*\alpha.$$

The map π_* *is called* integration over the fibre.

Proof. We recall the definition of $\pi_*\alpha$ for an ordinary differential form $\alpha \in \Omega^{n+k}_{\mathrm{vc}}(E)$. Given $x \in B$ and $v_1, \ldots, v_k \in T_xB$, choose lifts $V_1, \ldots, V_k : E_x \to TE$ of $v_1, \ldots, v_k$, respectively, and define

$$(\pi_*\alpha)_x(v_1, \ldots, v_k) := \int_{E_x} \iota(V_k) \cdots \iota(V_1)\alpha.$$

The integrand on the right (as an n-form on E_x) is independent of the choice of the lifts V_i. This defines a G-equivariant map $\pi_* : \Omega^*_{\mathrm{vc}}(E) \to \Omega^{*-n}(B)$. Hence it induces a map from $\Omega^*_{\mathrm{G,vc}}(E)$ to

$\Omega_{\mathrm{G}}^{*-n}(B)$. For $\xi \in \mathfrak{g}$ let $X_\xi \in \mathrm{Vect}(B)$ and $Y_\xi \in \mathrm{Vect}(E)$ denote the infinitesimal actions. Then Y_ξ is a lift of X_ξ and hence

$$\pi_* \iota(Y_\xi)\alpha = \iota(X_\xi)\pi_*\alpha$$

for every $\alpha \in \Omega_{\mathrm{G,vc}}^*(E)$. Moreover, it is shown in [4, Proposition 6.14.1] that

$$\pi_* \circ d = d \circ \pi_*.$$

This proves the *chain map* property of π_*. The *Thom class*, *module structure*, and *connection* properties are straightforward exercises. To prove *functoriality* we choose a local slice $(U_0, \phi_0, \mathrm{G}_0)$ of the G-action on B and assume that α is supported in $\pi^{-1}(\mathrm{G} \cdot \phi_0(U_0))$. Let $\Phi_0 : U_0 \times \mathbb{R}^n \to E$ be a G-equivariant trivialization of E along ϕ_0. Let $\mathrm{pr} : U_0 \times \mathbb{R}^n \to U_0$ denote the obvious projection and $A \in \Omega^1(B, \mathfrak{g})$ be a connection 1-form. Then, by the definition of the integral and Fubini's theorem,

$$\begin{aligned}
|\mathrm{G}_0| \int_{E/\mathrm{G}} \alpha &= \int_{U_0 \times \mathbb{R}^n} \Phi_0^* \alpha_{\pi^* A} \\
&= \int_{U_0} \mathrm{pr}_* \Phi_0^* \alpha_{\pi^* A} \\
&= \int_{U_0} \phi_0^* \pi_* \alpha_{\pi^* A} \\
&= \int_{U_0} \phi_0^* (\pi_* \alpha)_A \\
&= |\mathrm{G}_0| \int_{B/\mathrm{G}} \pi_* \alpha.
\end{aligned}$$

This proves the theorem. □

Remark 6.2. The equivariant Thom isomorphism theorem asserts that the map $\pi_* : \Omega_{\mathrm{G,vc}}^*(E) \to \Omega_{\mathrm{G}}^{*-n}(B)$ induces an isomorphism of cohomology whose inverse is induced by the map

$$\Omega_{\mathrm{G}}^{*-n}(B) \to \Omega_{\mathrm{G,vc}}^*(E) : \beta \mapsto \beta \wedge \tau$$

(see [2] and [13, Theorem 10.6.1]).

Corollary 6.3. *Suppose that* G *acts on* B *with finite isotropy and denote by* $\iota : B \to E$ *the inclusion of the zero section. Let* $\tau \in \Omega_{\mathrm{G,vc}}^n(E)$ *be an equivariant Thom form on* E *supported in an open neighbourhood* $U \subset E$ *of the zero section that intersects each fibre in a convex set. Then*

$$\int_{E/\mathrm{G}} \beta \wedge \tau = \int_{B/\mathrm{G}} \iota^* \beta$$

for every G-*closed form* $\beta \in \Omega_{\mathrm{G}}^*(E)$ *whose support intersects the closure of* U *in a compact set.*

Proof. The proof is an equivariant version of the proof of [1, Proposition 6.24]. We first observe that the form $\beta - \pi^* \iota^* \beta$ is G-exact. More precisely, there exists an equivariant differential form $\gamma \in \Omega_{\mathrm{G}}^*(E)$ such that

$$\beta = \pi^* \iota^* \beta + d_{\mathrm{G}} \gamma$$

and the support of γ intersects the closure of U in a compact set. To see this define $\phi_t : E \to E$ by

$$\phi_t(x,e) := (x, te)$$

and note that

$$\beta - \pi^*\iota^*\beta = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*\beta \,\mathrm{d}t.$$

Now compute

$$\begin{aligned}\int_{E/\mathrm{G}} \beta \wedge \tau &= \int_{E/\mathrm{G}} \pi^*\iota^*\beta \wedge \tau \\ &= \int_{B/\mathrm{G}} \pi_*(\pi^*\iota^*\beta \wedge \tau) \\ &= \int_{B/\mathrm{G}} \iota^*\beta \wedge \pi_*\tau \\ &= \int_{B/\mathrm{G}} \iota^*\beta.\end{aligned}$$

This proves the corollary. □

Corollary 6.4. *Suppose that* G *acts on B with finite isotropy and let $S : B \to E$ be a* G*-equivariant section which is transverse to the zero section. Then*

$$\int_{B/\mathrm{G}} \alpha \wedge S^*\tau = \int_{S^{-1}(0)/\mathrm{G}} \alpha$$

for every G*-closed form $\alpha \in \Omega^*_{\mathrm{G}}(B)$ whose support intersects the closure of $S^{-1}(U)$ in a compact set.*

Proof. By Theorem 5.3, we may assume, without loss of generality, that the support of the pullback $S^*\tau$ is contained in a tubular neighbourhood N of $S^{-1}(0)$. Since the image of a fibre of the normal bundle under S is homotopic to a fibre of E the integral of $S^*\tau$ over each fibre of the normal bundle is one. Hence $S^*\tau$ is a Thom form on the normal bundle of $S^{-1}(0)$ and so the result follows from Corollary 6.3. □

Corollary 6.5. *Let $E \to B$ be a complex vector bundle equipped with the standard S^1-action over a compact manifold B (on which S^1 acts trivially) and denote by $\iota : B \to E$ the inclusion of the zero section. Suppose $\tau \in \Omega^*_{S^1}(E)$ is an equivariant Thom form. Then*

$$\iota^*\tau(\eta) = \sum_{j=0}^{\operatorname{rank} E} \left(\frac{\mathrm{i}\eta}{2\pi}\right)^{\operatorname{rank} E - j} \tau_j,$$

where $\tau_j \in \Omega^{2j}(B)$ is a closed form representing the jth Chern class $c_j(E)$.

Proof. For $j = n := \operatorname{rank} E$ it follows from Corollary 6.4 the fact that τ_n is a (nonequivariant) Thom form on E and the fact that $c_n(E)$ is the Euler class. For the trivial bundle $E = B \times \mathbb{C}^n$ the result follows by considering the Thom form

$$\tau(\eta) = \sum_{k=0}^{n} (\mathrm{i}\eta)^{n-k} f_{n-k}(|z|^2/2) \frac{\omega^k}{k!},$$

where $\omega \in \Omega^2(\mathbb{C}^n)$ is the standard symplectic form and the functions f_k are as in the proof of Proposition 5.4. The result then follows from the fact that $f_n(0) = 1/2^{n-1}(n-1)!\mathrm{Vol}(S^{2n-1}) = (2\pi)^{-n}$. If $\dim M = 2k < \operatorname{rank} E$ then, for $j = k$, the result follows by splitting E into a bundle of rank k and the trivial bundle. To prove the result in general, consider the pullbacks of E under all smooth maps $f : X \to M$, defined on compact manifolds of dimension $2j$. □

7. Finite dimensional reduction

In Section 5 we have defined the equivariant Euler class for oriented regular finite dimensional G-moduli problems. In the following two sections we explain how to extend the definition to the infinite dimensional (and the nonorientable finite dimensional) case by means of finite dimensional reduction. The first step is to show that the Euler class of oriented regular finite dimensional G-moduli problems satisfies the (*Functoriality*) axiom.

Proposition 7.1. *Let (B_0, E_0, S_0) and (B_1, E_1, S_1) be oriented regular finite dimensional* G-*moduli problems and let (ψ, Ψ) be a morphism from (B_0, E_0, S_0) to (B_1, E_1, S_1). Then*

$$\chi^{B_0,E_0,S_0}(\psi^*\alpha_1) = \chi^{B_1,E_1,S_1}(\alpha_1)$$

for every G-*closed equivariant differential form $\alpha_1 \in \Omega^*_{\mathrm{G}}(B_1)$.*

Proof. Shrinking B_0, if necessary, we may assume that the embedding ψ of a neighbourhood of $M_0 = S_0^{-1}(0) \subset B_0$ into B_1 is defined on all of B_0. Choose a G-invariant splitting

$$E_1 = E_{10} \oplus E_{11}$$

near $\psi(B_0)$ such that E_{10} agrees with the image of the inclusion $\Psi : E_0 \to E_1$ over $\psi(B_0)$. Then the section $S_1 : B_1 \to E_1$ can be written as

$$S_1 = S_{10} \oplus S_{11}.$$

Note that Ψ identifies the G-moduli problem (B_0, E_0, S_0) with the restriction $(\psi(B_0), E_{10}, S_{10})$.

We prove that S_{11} is transverse to the zero section near $M_1 = \psi(M_0)$ and that the kernel of $DS_{11}(\psi(x))$ agrees with the image of $\mathrm{d}\psi(x)$ for x near $M_0 = S_0^{-1}(0)$. Surjectivity of $DS_{11}(\psi(x))$ for $x \in M_0$ follows from (1):

$$E_{1\psi(x)} = (\operatorname{im} DS_{10}(\psi(x)) \oplus \Psi_x \operatorname{coker} DS_0(x)) \oplus \operatorname{im} DS_{11}(\psi(x)).$$

To prove the second assertion note that the indices of S_0 and S_1 agree and hence $\operatorname{rank} E_{11} = \operatorname{rank} E_1 - \operatorname{rank} E_0 = \dim B_1 - \dim B_0$. Moreover, S_{11} vanishes over $\psi(B_0)$ and so $\operatorname{im} \mathrm{d}\psi(x) \subset \ker DS_{11}(\psi(x))$ for every $x \in B_0$, with equality if and only if $DS_{11}(\psi(x))$ is surjective. Hence, for $x \in M_0$, we have

$\ker DS_{11}(\psi(x)) = \operatorname{im} d\psi(x)$. This proves the claim. Shrinking B_0 and B_1, if necessary, we may assume that $\psi(B_0) = S_{11}^{-1}(0)$ and that S_{11} is transverse to the zero section.

Choose an equivariant Thom form

$$\tau_1 = \tau_{10} \wedge \tau_{11}$$

on E_1 such that τ_{10} is a Thom form for E_{10} and τ_{11} is a Thom form for E_{11}. Choose a tubular neighbourhood $U_1 \subset B_1$ of $\psi(B_0)$ such that $S_{11}^*\tau_{11} \in \Omega_G^*(B_1)$ is supported in U_1. Then, by Corollary 6.4,

$$\int_{B_1/G} \beta \wedge S_{11}^*\tau_{11} = \int_{B_0/G} \psi^*\beta$$

for every G-closed form $\beta \in \Omega_G^*(B_1)$ whose support intersects the closure of U_1 in a compact set. Moreover, $\tau_0 := \Psi^*\tau_{10}$ is a Thom form on E_0. Hence

$$\begin{aligned}\int_{B_1/G} \alpha_1 \wedge S_1^*\tau_1 &= \int_{B_1/G} \alpha_1 \wedge S_{10}^*\tau_{10} \wedge S_{11}^*\tau_{11}\\ &= \int_{B_0/G} \psi^*(\alpha_1 \wedge S_{10}^*\tau_{10})\\ &= \int_{B_0/G} \psi^*\alpha_1 \wedge S_0^*\Psi^*\tau_{10}\\ &= \int_{B_0/G} \psi^*\alpha_1 \wedge S_0^*\tau_0.\end{aligned}$$

This proves the proposition. □

An example of a morphism is the inclusion of a G-moduli problem into its stabilization by a G-representation V.

Definition 7.2. Let V be a real Hilbert space with an orthogonal action of G and $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ be a G-moduli problem. The G-moduli problem $(\mathscr{B}^V, \mathscr{E}^V, \mathscr{S}^V)$ defined by

$$\mathscr{B}^V := \mathscr{B} \times V, \qquad \mathscr{E}^V_{x,v} := \mathscr{E}_x \times V, \qquad \mathscr{S}^V(x,v) := (\mathscr{S}(x), v)$$

is called the *stabilization of* $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ *by* V. The morphism (ψ, Ψ) from $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ to $(\mathscr{B}^V, \mathscr{E}^V, \mathscr{S}^V)$, given by

$$\psi(x) := (x, 0), \qquad \Psi_x e := (e, 0),$$

is called the *stabilization morphism*.

Definition 7.3. (i) Let $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ be a G-moduli problem *A finite dimensional reduction* of $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ is a sixtuple $R = (B, E, S, V, \psi, \Psi)$ such that (B, E, S) is an oriented finite dimensional G-moduli problem, V is a finite dimensional real Hilbert space with an orthogonal linear G-action, and (ψ, Ψ) is a morphism from (B, E, S) to $(\mathscr{B}^V, \mathscr{E}^V, \mathscr{S}^V)$.

(ii) Let $R_0 = (B_0, E_0, S_0, V_0, \psi_0, \Psi_0)$ and $R_1 = (B_1, E_1, S_1, V_1, \psi_1, \Psi_1)$ be two finite dimensional reductions of $(\mathscr{B}, \mathscr{E}, \mathscr{S})$. A *morphism* (*of finite dimensional reductions*) from R_0 to R_1 is a triple (ψ, Ψ, T), where (ψ, Ψ) is a morphism from (B_0, E_0, S_0) to (B_1, E_1, S_1), $T : V_0 \to V_1$ is a G-equivariant injective linear map, and the following diagram commutes:

$$\begin{array}{ccc} (B_0, E_0, S_0) & \xrightarrow{(\psi_0, \Psi_0)} & (\mathscr{B}^{V_0}, \mathscr{E}^{V_0}, \mathscr{S}^{V_0}) \\ {\scriptstyle (\psi,\Psi)}\downarrow & & \downarrow{\scriptstyle T} \\ (B_1, E_1, S_1) & \xrightarrow{(\psi_1, \Psi_1)} & (\mathscr{B}^{V_1}, \mathscr{E}^{V_1}, \mathscr{S}^{V_1}) \end{array}.$$

We write $R_0 \preccurlyeq R_1$ if there exists a morphism (ψ, Ψ, T) from R_0 to R_1. Two finite dimensional reductions R_0 and R_1 are called *equivalent* if $R_0 \preccurlyeq R_1$ and $R_1 \preccurlyeq R_0$.

The main results of this section assert that finite dimensional reductions exist and form a directed system.

Theorem 7.4. *Every* G-*moduli problem* $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ *admits a finite dimensional reduction.*

Theorem 7.5. *If* R_0, R_1 *are finite dimensional reductions of* $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ *then there exists a finite dimensional reduction* R *such that* $R_0 \preccurlyeq R$ *and* $R_1 \preccurlyeq R$.

The proofs are based on the existence of *families of complements*.

Definition 7.6. A *family of complements* for $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ is a pair (V, Γ) such that V is an oriented finite dimensional real Hilbert space equipped with an orthogonal linear G-action,

$$\Gamma : \mathscr{B} \times V \to \mathscr{E}$$

is a G-equivariant bundle homomorphism, and

$$\mathscr{E}_x = \operatorname{im} \mathscr{D}_x + \operatorname{im} \Gamma_x$$

for every $x \in \mathscr{M} = \mathscr{S}^{-1}(0)$, where $\mathscr{D}_x := D\mathscr{S}(x)$ denotes the vertical differential of $\mathscr{S}$.

Proposition 7.7. *Let* (V, Γ) *be a family of complements for* $(\mathscr{B}, \mathscr{E}, \mathscr{S})$. *Then there exists a neighbourhood* $\mathscr{U} \subset \mathscr{B}$ *of* $\mathscr{M}$ *and a* $\delta > 0$ *such that the sixtuple* $R^\Gamma := (B^\Gamma, E^\Gamma, S^\Gamma, V, \psi^\Gamma, \Psi^\Gamma)$, *defined by*

$$B^\Gamma := \{(x, v) \in \mathscr{U} \times V \mid \mathscr{S}(x) = \Gamma_x v,\ |v| < \delta\}, \quad E^\Gamma_{(x,v)} := V,$$

$$S^\Gamma(x, v) := v, \qquad \psi^\Gamma(x, v) := (x, v), \qquad \Psi^\Gamma_{(x,v)} w := (\Gamma_x w, w),$$

is a finite dimensional reduction of $(\mathscr{B}, \mathscr{E}, \mathscr{S})$.

Proof. Γ is transverse to $\mathscr{S}$ at every point $(x, 0) \in \mathscr{M} \times V$. Hence there exists a neighbourhood $\mathscr{U} \subset \mathscr{B}$ of $\mathscr{M}$ and a $\delta > 0$ such that Γ is transverse to $\mathscr{S}$ at every point $(x, v) \in \mathscr{U} \times V$ such that $|v| < \delta$. It follows that B^Γ is a submanifold of $\mathscr{B} \times V$ of dimension

$$\dim B^\Gamma = \operatorname{index}(\mathscr{S}) + \dim \mathrm{G} + \dim V.$$

Hence every section of $E^\Gamma = B^\Gamma \times V$ has the same index as $\mathscr{S}$. We prove that $\mathscr{S}^V \circ \psi^\Gamma = \Psi^\Gamma \circ S^\Gamma$:

$$\mathscr{S}^V(\psi^\Gamma(x,v)) = \mathscr{S}^V(x,v) = (\mathscr{S}(x),v) = (\Gamma_x v, v) = \Psi^\Gamma_{(x,v)} v = \Psi^\Gamma_{(x,v)} S^\Gamma(x,v).$$

The zero set of S^Γ is $M^\Gamma = \{(x,0) \mid x \in \mathscr{M}\}$ and so $\iota^\Gamma(M^\Gamma) = \mathscr{M} \times \{0\} = \mathscr{M}^V$. Next we observe that the tangent space of B^Γ at the point $(x,0)$ is given by

$$T_{(x,0)}B^\Gamma = \{(\hat{x},\hat{v}) \in T_x\mathscr{B} \times V \mid \mathscr{D}_x\hat{x} = \Gamma_x\hat{v}\}.$$

The image of this space under the differential of inclusion $\psi^\Gamma : B^\Gamma \to \mathscr{B} \times V$ contains the kernel of the operator $\mathscr{D}^V_{(x,0)} : T_x\mathscr{B} \times V \to \mathscr{E}_x \times V$. Since

$$\operatorname{im} \mathscr{D}^V_{(x,0)} = \{(\mathscr{D}_x\hat{x}, \hat{v}) \mid \hat{x} \in T_x\mathscr{B},\ \hat{v} \in V\},$$

$$\operatorname{im} \Psi^\Gamma_{(x,0)} = \{(\Gamma_x w, w) \mid w \in V\},$$

we obtain $\operatorname{im} \mathscr{D}^V_{(x,0)} + \operatorname{im} \Psi^\Gamma_{(x,0)} = \mathscr{E}^V_{(x,0)}$ for every $x \in \mathscr{M}$.

We prove that B^Γ is oriented. Since $\mathscr{S} - \Gamma$ is transverse to the zero section, it suffices to show that $\det(\mathscr{S} - \Gamma) \cong \det(\mathscr{S})$. This follows from a standard argument for determinant line bundles: If $\mathscr{X}$ and $\mathscr{Y}$ are Hilbert spaces, $\mathscr{D} : \mathscr{X} \to \mathscr{Y}$ is a Fredholm operator, V is a finite dimensional oriented Hilbert space, and $\Gamma : V \to \mathscr{Y}$ is a linear operator, then there is a canonical isomorphism

$$\det(\mathscr{D} - \Gamma) \cong \det(\mathscr{D}).$$

Here the operator $\mathscr{D} - \Gamma : \mathscr{X} \oplus V \to \mathscr{Y}$ is given by

$$(\mathscr{D} - \Gamma)(x,v) := \mathscr{D}x - \Gamma v.$$

To see this consider the exact sequence

$$0 \to \ker \mathscr{D} \oplus \ker \Gamma \to \ker(\mathscr{D} - \Gamma) \to \operatorname{im} \mathscr{D} \cap \operatorname{im} \Gamma.$$

It shows that there is a canonical isomorphism

$$\Lambda^{\max} \ker(\mathscr{D} - \Gamma) \cong \Lambda^{\max} \ker \mathscr{D} \otimes \Lambda^{\max} \ker \Gamma \otimes \Lambda^{\max}(\operatorname{im} \mathscr{D} \cap \operatorname{im} \Gamma). \tag{18}$$

Since $\operatorname{im} \Gamma/(\operatorname{im} \Gamma \cap \operatorname{im} \mathscr{D}) \cong \operatorname{im}(\mathscr{D} - \Gamma)/\operatorname{im} \mathscr{D}$ we have

$$\begin{aligned}\Lambda^{\max} \operatorname{coker} \mathscr{D} &\cong \Lambda^{\max} \operatorname{coker}(\mathscr{D} - \Gamma) \otimes \Lambda^{\max} \left(\frac{\operatorname{im}(\mathscr{D} - \Gamma)}{\operatorname{im} \mathscr{D}}\right) \\ &\cong \Lambda^{\max} \operatorname{coker}(\mathscr{D} - \Gamma) \otimes \Lambda^{\max} \left(\frac{\operatorname{im} \Gamma}{\operatorname{im} \Gamma \cap \operatorname{im} \mathscr{D}}\right)\end{aligned}$$

and hence

$$\begin{aligned}\Lambda^{\max}(\operatorname{im} \mathscr{D} \cap \operatorname{im} \Gamma) &\cong \Lambda^{\max} \operatorname{im} \Gamma \otimes \Lambda^{\max} \left(\frac{\operatorname{im} \Gamma}{\operatorname{im} \mathscr{D} \cap \operatorname{im} \Gamma}\right)^* \\ &\cong \Lambda^{\max} \operatorname{im} \Gamma \otimes \Lambda^{\max} \operatorname{coker}(\mathscr{D} - \Gamma) \otimes \Lambda^{\max}(\operatorname{coker} \mathscr{D})^*.\end{aligned}$$

Inserting this identity into (18) and using $\Lambda^{\max} \ker \Gamma \otimes \Lambda^{\max} \operatorname{im} \Gamma \cong \Lambda^{\max} V \cong \mathbb{R}$, we find

$$\det(\mathscr{D} - \Gamma) \cong \Lambda^{\max} \ker \mathscr{D} \otimes \Lambda^{\max}(\operatorname{coker} \mathscr{D})^* = \det(\mathscr{D})$$

as claimed. □

Proof of Theorem 7.4. By Proposition 7.7, it suffices to prove the existence of a family of complements (V,Γ). Let $x_0 \in \mathscr{M} = \mathscr{S}^{-1}(0)$, denote by $\mathrm{G}_0 \subset \mathrm{G}$ the stabilizer of x_0, and let $E_0 \subset \mathscr{E}_{x_0}$ denote the orthogonal complement of the image of $\mathscr{D}_{x_0}$. By the Fredholm property, E_0 is a finite dimensional vector space. The group G_0 acts on $T_{x_0}\mathscr{B}$ and $\mathscr{E}_{x_0}$, and the operator $\mathscr{D}_{x_0} : T_{x_0}\mathscr{B} \to \mathscr{E}_{x_0}$ is G_0-equivariant (because $\mathscr{S}$ is G-equivariant). Hence E_0 inherits an orthogonal linear action of G_0. Consider the infinite dimensional vector space

$$\mathscr{V}_0 := \{v \in C^\infty(\mathrm{G}, E_0) \mid v(hg_0) = g_0^* v(h)\ \forall h \in \mathrm{G}\ \forall g_0 \in \mathrm{G}_0\}.$$

The group G acts on $\mathscr{V}_0$ by

$$(gv)(h) := v(g^{-1}h) \tag{19}$$

for $g, h \in \mathrm{G}$.

We prove that there exists a finite dimensional G-invariant subspace $V_0 \subset \mathscr{V}_0$ such that

$$E_0 = \{v_0(\mathbb{1}) \mid v_0 \in V_0\}.$$

To see this, choose any basis $e_1, \dots, e_m$ of E_0 and choose sections $v_i \in \mathscr{V}_0$ such that $v_i(\mathbb{1}) = e_i$. Choose $\varepsilon > 0$ such that the vectors $v_i'(\mathbb{1}), \dots, v_m'(\mathbb{1})$ are linearly independent whenever $v_1', \dots, v_m' \in \mathscr{V}_0$ such that

$$\|v_i' - v_i\|_{L^\infty} < \varepsilon.$$

Now the eigenspaces of the Laplace operator

$$\Delta = d^*d : \mathscr{V}_0 \to \mathscr{V}_0,$$

with respect to a biinvariant metric on G, are G-invariant and finite dimensional. Moreover, every element of $\mathscr{V}_0$ can be approximated in the L^∞ norm by finite linear combinations of eigenfunctions. Hence the functions $v_i' \in \mathscr{V}_0$ can be chosen such that each v_i' is contained in a finite dimensional G-invariant subspace $V_i \subset \mathscr{V}_0$. The subspace

$$V_0 := V_1 + \cdots + V_m$$

has the required properties. (The subspaces V_i can also be obtained as a consequence of the Peter–Weyl Theorem [5, Theorem 5.7].)

Now let $K : \mathscr{B} \times V_0 \to \mathscr{E}$ be any bundle homomorphism such that

$$K_{g_*x_0} v_0 = g_* v_0(g) \in \mathscr{E}_{g_*x_0}$$

for $g \in \mathrm{G}$ and $v_0 \in V_0$, where $g_*x := (g^{-1})^*x$. To see that such a homomorphism exists note first that, since $v_0(hg_0) = g_0^* v_0(h)$, the homomorphism $K_x : V_0 \to \mathscr{E}_x$ is well defined for $x \in \mathrm{G}_*x_0 := \{g_*x_0 \mid g \in \mathrm{G}\}$. Secondly, since G_*x_0 is a submanifold of $\mathscr{B}$, K can be extended by a partition of unity construction (see [15, p. 30] for partitions of unity on Hilbert manifolds) to a homomorphism from $\mathscr{B} \times V_0$ to $\mathscr{E}$. The resulting homomorphism is not necessarily G-equivariant. Define $\Gamma_0 : \mathscr{B} \times V_0 \to \mathscr{E}$ by

$$\Gamma_{0x} v_0 := \frac{1}{\mathrm{Vol(G)}} \int_{\mathrm{G}} g^* K_{g_*x} g v_0 \,\mathrm{d}g \in \mathscr{E}_x$$

for $x \in \mathscr{B}$ and $v_0 \in V_0$, where $gv_0 \in V_0$ is given by (19). Then Γ_0 is G-equivariant and $\Gamma_{0x_0} v_0 = v_0(\mathbb{1})$.

Now cover the compact set $\mathscr{M} \subset \mathscr{B}$ by finitely many open sets $\mathscr{U}_1, \dots, \mathscr{U}_N$ such that, for each $i \in \{1, \dots, N\}$, there exists a G-equivariant homomorphism $\Gamma_i : \mathscr{B} \times V_i \to \mathscr{E}$ such that

$$\operatorname{im} \mathscr{D}_x + \operatorname{im} \Gamma_{ix} = \mathscr{E}_x$$

for $x \in \mathscr{U}_i$. Define

$$V := V_1 \oplus \cdots \oplus V_N$$

and $\Gamma_x : V \to \mathscr{E}_x$ by

$$\Gamma_x(v_1, \dots, v_N) := \Gamma_{1x} v_1 + \cdots + \Gamma_{Nx} v_N.$$

Then (V, Γ) is a family of complements. □

Proof of Theorem 7.5. The proof has three steps.

Step 1: *For every finite dimensional reduction R of $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ there exists a finite dimensional reduction $R' = (B', E', S', V', \psi', \Psi')$ such that $R \preccurlyeq R'$ and the bundle $E' \to B'$ admits a trivialization.*

Let $R = (B, E, S, V, \psi, \Psi)$. Shrinking B, if necessary, we may assume that there exists a finite dimensional Hilbert space W equipped with an orthogonal linear G-action and an injective G-equivariant vector bundle homomorphism $E \to B \times W : (x, e) \mapsto (x, \Phi_x e)$. Define R' by

$$B' := \{(x, w) \in B \times W \mid w \perp \operatorname{im} \Phi_x\}, \quad \psi'(x, w) := (\psi(x), w),$$

$$E' := B' \times W, \quad \Psi'_{(x,w)}(\Phi_x e + w_1) := (\Psi_x e, w_1),$$

$$V' := V \times W, \quad S'(x, w) := \Phi_x S(x) + w$$

for $x \in B$, $e \in E_x$, and $w, w_1 \in (\operatorname{im} \Phi_x)^\perp$. Then $R \preccurlyeq R'$.

Step 2: *For every finite dimensional reduction $R = (B, E, S, V, \psi, \Psi)$ of a* G-*moduli problem $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ there exists a family of complements (W, Γ) for $(\mathscr{B}^V, \mathscr{E}^V, \mathscr{S}^V)$ such that $R \preccurlyeq R^\Gamma$.*

By Step 1, we may assume without loss of generality that $E = B \times W$. Choose any bundle homomorphism $\Gamma : \mathscr{B}^V \times W \to \mathscr{E}^V$ such that

$$\Gamma_{\psi(x)} = \Psi_x : W \to \mathscr{E}^V_{\psi(x)}$$

for x near $M = S^{-1}(0) \subset B$. Then $R \preccurlyeq R^\Gamma$. Note, in particular, that

$$B^\Gamma = \{(x, v, w) \in \mathscr{B} \times V \times W \mid \Gamma_{x,v} w = \mathscr{S}^V(x, v)\}.$$

The inclusion $B \to B^\Gamma$ is given by $x \mapsto (\psi(x), S(x))$ and the bundle homomorphism $E = B \times W \to E^\Gamma = B^\Gamma \times W$ is the obvious lift of this inclusion.

Step 3: *We prove Theorem* 7.5.

By Step 2, we may assume that $R_0 = R^{\Gamma_0}$ and $R_1 = R^{\Gamma_1}$ for two families of complements (V_0, Γ_0) and (V_1, Γ_1). Define a family of complements (V, Γ) by

$$V := V_0 \oplus V_1, \qquad \Gamma_x(v_0, v_1) := \Gamma_{0x} v_0 + \Gamma_{1x} v_1$$

for $x \in \mathscr{B}$, $v_0 \in V_0$, and $v_1 \in V_1$. Then $R^{\Gamma_i} \preccurlyeq R^\Gamma$ for $i = 0, 1$. □

8. Construction of the Euler class

Let $(\mathscr{B},\mathscr{E},\mathscr{S})$ be a regular G-moduli problem. We define the *Euler class*

$$\chi^{\mathscr{B},\mathscr{E},\mathscr{S}} : H^*_{\mathrm{G}}(\mathscr{B};\mathbb{R}) \to \mathbb{R}$$

as follows. Let $\alpha \in \Omega^*_{\mathrm{G}}(\mathscr{B})$ be equivariantly closed and $R = (B, E, S, V, \psi, \Psi)$ be a finite dimensional reduction of $(\mathscr{B},\mathscr{E},\mathscr{S})$. Let (U, τ) be a Thom structure on (B, E, S). We define

$$\chi^{\mathscr{B},\mathscr{E},\mathscr{S}}(\alpha) := \chi^{B,E,S}(\psi^*\alpha^V) := \int_{B/\mathrm{G}} \psi^*\alpha^V \wedge S^*\tau, \tag{20}$$

where $\alpha^V \in \Omega^*_{\mathrm{G}}(\mathscr{B}^V)$ is the pullback of $\alpha \in \Omega^*_{\mathrm{G}}(\mathscr{B})$ under the obvious G-equivariant projection $\mathscr{B}^V = \mathscr{B} \times V \to \mathscr{B}$. Since the difference of two Thom forms is exact, the integral in (20) is independent of the choice of the Thom structure. Since τ is G-closed it depends only on the equivariant cohomology class of α.

Proposition 8.1. *The Euler class $\chi^{\mathscr{B},\mathscr{E},\mathscr{S}}$ is independent of the finite dimensional reduction R used to define it. It satisfies, and is uniquely determined by, the* (*Functoriality*) *and* (*Thom class*) *axioms.*

Proof. See Proposition 7.1 and Theorem 7.5. □

Proposition 8.2. *The Euler class satisfies the* (*Transversality*) *axiom.*

Proof. Suppose $\mathscr{S}$ is transverse to the zero section and let (B, E, S) be a finite dimensional reduction of $(\mathscr{B},\mathscr{E},\mathscr{S})$. Then S is also transverse to the zero section. Hence the (*Transversality*) axiom follows from Corollary 6.4. □

Proposition 8.3. *The Euler class satisfies the* (*Cobordism*) *axiom.*

Proof. Let $\tilde{\mathscr{E}} \to \tilde{\mathscr{B}}$ be a G-equivariant Hilbert space bundle over a Hilbert manifold with boundary $\mathscr{B} := \partial\tilde{\mathscr{B}}$ and $\tilde{\mathscr{S}} : \tilde{\mathscr{B}} \to \tilde{\mathscr{E}}$ be an oriented Fredholm section with compact zero set. Suppose G acts with finite isotropy on $\tilde{\mathscr{B}}$. Denote by $\iota : \mathscr{B} \to \tilde{\mathscr{B}}$ the inclusion of the boundary and by $(\mathscr{B},\mathscr{E},\mathscr{S})$ the restriction of $(\tilde{\mathscr{B}},\tilde{\mathscr{E}},\tilde{\mathscr{S}})$ to the boundary. We must prove that

$$\chi^{\mathscr{B},\mathscr{E},\mathscr{S}}(\iota^*\tilde{\alpha}) = 0$$

for every $\tilde{\alpha} \in H^*_{\mathrm{G}}(\tilde{\mathscr{B}})$. To see this note that the proofs of Theorem 7.4 and Proposition 7.7 carry over to G-moduli problems with boundary. Hence assume that $\Gamma : \tilde{\mathscr{B}} \times V \to \tilde{\mathscr{E}}$ is a family of complements for $(\tilde{\mathscr{B}},\tilde{\mathscr{E}},\tilde{\mathscr{S}})$. Then, as in the proof of Proposition 7.7, there exist an open neighbourhood $\tilde{\mathscr{U}} \subset \tilde{\mathscr{B}}$ of $\tilde{\mathscr{M}} = \tilde{\mathscr{S}}^{-1}(0)$ and a $\delta > 0$ such that Γ is transverse to $\tilde{\mathscr{S}}$ at every point $(\tilde{x}, v) \in \tilde{\mathscr{U}} \times V$ such that $|v| < \delta$. Define

$$\tilde{B} := \{(\tilde{x}, v) \in \tilde{\mathscr{U}} \times V \mid \tilde{\mathscr{S}}(\tilde{x}) = \Gamma_{\tilde{x}} v,\ |v| < \delta\}$$

and

$$B := \tilde{B} \cap (\mathscr{B} \times V).$$

Then $\tilde{B}$ is a smooth finite dimensional manifold with boundary $\partial\tilde{B} = B$. Consider the section $\tilde{S} : \tilde{B} \to V$ defined by

$$\tilde{S}(x, v) := v$$

and let $S : B \to V$ denote its restriction to the boundary. By Proposition 7.7, the triple (B, E, S) with $E := B \times V$ is a finite dimensional reduction of $(\mathscr{B}, \mathscr{E}, \mathscr{S})$. Let $\tilde{\psi} : \tilde{B} \to \tilde{\mathscr{B}}$ and $\psi : B \to \mathscr{B}$ be defined by

$$\tilde{\psi}(\tilde{x}, v) := \tilde{x}, \qquad \psi(x, v) := x.$$

Then ψ is the restriction of $\tilde{\psi}$ to the boundary. Let $\tilde{\tau} \in \Omega^*_{\mathrm{G}}(\tilde{B} \times V)$ be an equivariant Thom form and denote by τ its restriction to $B \times V$. Then for every $\tilde{\alpha} \in \Omega^*_{\mathrm{G}}(\tilde{\mathscr{B}})$ the form $\psi^* \iota^* \tilde{\alpha} \wedge S^* \tau \in \Omega^*_{\mathrm{G}}(B)$ is the restriction of the form $\tilde{\psi}^* \tilde{\alpha} \wedge \tilde{S}^* \tilde{\tau} \in \Omega^*_{\mathrm{G}}(\tilde{B})$ to the boundary. Hence

$$\begin{aligned}
\chi^{\mathscr{B},\mathscr{E},\mathscr{S}}(\iota^* \tilde{\alpha}) &= \chi^{B,E,S}(\psi^* \iota^* \tilde{\alpha}) \\
&= \int_{B/\mathrm{G}} \psi^* \iota^* \tilde{\alpha} \wedge S^* \tau \\
&= \int_{\tilde{B}/\mathrm{G}} d_{\mathrm{G}}(\tilde{\psi}^* \tilde{\alpha} \wedge \tilde{S}^* \tilde{\tau}) \\
&= 0.
\end{aligned}$$

The penultimate equation follows from Proposition 4.2(ii) and the last equation follows from the fact that $\tilde{\alpha}$ and $\tilde{\tau}$ are equivariantly closed. □

Proposition 8.4. *The Euler class satisfies the* (*Subgroup*) *axiom.*

Proof. Let B be a (finite dimensional) manifold with a smooth G action with finite isotropy and suppose that $\mathrm{H} \subset \mathrm{G}$ is a normal subgroup that acts freely on B. Denote $\mathfrak{h} := \mathrm{Lie}(\mathrm{H})$ and let $\pi : B \to B/\mathrm{H}$ be the obvious projection. Let $A \in \Omega^1(B, \mathfrak{g})$ be a connection 1-form and denote by $\pi_* A \in \Omega^1(B/\mathrm{H}, \mathfrak{g}/\mathfrak{h})$ the induced connection 1-form on B/H. Then every local slice $\phi_0 : U_0 \to B$ determines a local slice $\pi \circ \phi_0 : U_0 \to B/\mathrm{H}$ for the G/H action on B/H. Now let $\alpha \in \Omega^*_{\mathrm{G/H}}(B/\mathrm{H})$ be a G/H-closed equivariant differential form, supported in $(\mathrm{G/H})^* \pi \circ \phi_0(U_0)$. Then the composition of $\alpha : \mathfrak{g}/\mathfrak{h} \to \Omega^*(B/\mathrm{H})$ with the projection $j : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$, followed by the pullback $\pi^* : \Omega^*(B/\mathrm{H}) \to \Omega^*(B)$, is a G-closed equivariant differential form on B which we denote by $\pi^* j^* \alpha \in \Omega^*_{\mathrm{G}}(B)$. It is supported in $\mathrm{G}^* \phi_0(U_0)$ and satisfies

$$(\pi^* j^* \alpha)_A = \pi^* \alpha_{\pi_* A}.$$

Hence

$$\int_{B/\mathrm{G}} \pi^* j^* \alpha = \int_{U_0} \phi_0^* (\pi^* j^* \alpha)_A = \int_{U_0} (\pi \circ \phi_0)^* \alpha_{\pi_* A} = \int_{(B/\mathrm{H})/(\mathrm{G/H})} \alpha.$$

This proves the proposition. □

We have established all properties of the Euler class except for the (*Rationality*) axiom. The proof relies upon an alternative construction of the Euler class via multivalued perturbations. After some preparations on weighted branched submanifolds the (*Rationality*) axiom is proved at the end of Section 10.

9. Weighted branched submanifolds

To prove the (*Rationality*) axiom it suffices, by Theorem 7.4, to consider the finite dimensional case. Let (B, E, S) be a finite dimensional G-moduli problem. In general, there is no G-equivariant perturbation of S which is transverse to the zero section. However, it is always possible to construct a multivalued perturbation $\Sigma : B \to 2^E$ with rational weights which is both equivariant and transverse to the zero section. This gives rise to an alternative definition of the function $\chi^{B,E,S}$ and shows that it takes rational values on $H^*_{\mathrm{G}}(\mathscr{B}; \mathbb{Q})$. Such multivalued perturbations were used by Fukaya and Ono [11] in their construction of the Gromov–Witten invariants on general symplectic manifolds. The following exposition grew out of discussions of the third author with Hofer in our attempt to understand Floer homology for general symplectic manifolds. A preliminary discussion of multivalued perturbations and branched manifolds can also be found in [23].

We begin with an exposition of *weighted branched submanifolds*. They will appear in the next section as zero sets of multivalued sections.

Definition 9.1. Let B be a finite dimensional manifold and G be a compact oriented Lie group which acts on B with finite isotropy. Let d be a nonnegative integer. A *weighted branched d-submanifold of B* is a function

$$\lambda : B \to \mathbb{Q} \cap [0, \infty)$$

with the following properties.

(**Equivariance**) $\lambda(g^* x) = \lambda(x)$ for all $x \in B$ and $g \in \mathrm{G}$.

(**Local structure**) For each $x_0 \in B$ there exist an open neighbourhood U of x_0, finitely many $(d + \dim \mathrm{G})$-submanifolds $M_1, \ldots, M_m \subset U$ (called *branches* of λ), and finitely many positive rational numbers $\lambda_1, \ldots, \lambda_m$ (called *weights*) such that each M_i is a relatively closed subset of U and

$$\lambda(x) = \sum_{x \in M_i} \lambda_i$$

for every $x \in U$.

A weighted branched d-submanifold λ of B is called *compact* if its *support*

$$M := \{x \in B \mid \lambda(x) > 0\}$$

is compact. A point $x \in M$ is called a *branch point* if the restriction of λ to M is not locally constant near x. The set of branch points will be denoted by M^{b}.

Remark 9.2. Note that d denotes the dimension of the quotient by G. An ordinary submanifold $M \subset B$ can be viewed as a weighted branched submanifold by taking for λ the characteristic function of M.

Remark 9.3. A point x is a branch point if and only if there exist two local branches M_i and M_j near x such that $x \in M_i \cap M_j \setminus \mathrm{int}_{M_i}(M_i \cap M_j)$. An intrinsic definition of branched manifold is given in [23, Definition 5.6]. As part of that definition it is required that

$$\mathrm{int}_{M_i}(M_i \cap M_j) = \mathrm{int}_{M_j}(M_i \cap M_j)$$

for any two local branches in U. This condition is automatically satisfied when M_i and M_j are submanifolds of B of the same dimension. Under this hypothesis it is proved in [23, Lemma 5.10] that the set of branch points is nowhere dense in M.

Example 9.4. Consider the branched 1-submanifold of the plane whose support is the union M of an embedded circle of length one and the graph of a smooth nonnegative function on the circle that vanishes on a Cantor set. Then the set M^{b} of branch points is the Cantor set. Its measure can be chosen arbitrarily close to one.

Example 9.5. The S-figure in a circle in the plane is not the support of a weighted branched 1-submanifold.

Example 9.6. This example shows that it is not always possible to choose the neighbourhood U in the local structure axiom to be G-invariant.

Let $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ act on $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ by

$$e^{i\theta}(z,w) := (e^{ik\theta}z, e^{i\ell\theta}w),$$

where k and ℓ are relatively prime. Then the subset

$$M := \{(z,w) \mid \mathrm{Re}(z^{\ell}\bar{w}^{k}) = 0\}$$

of S^3 is an S^1-invariant immersed 2-torus with transverse self-intersections. It is the support of a weighted branched 1-submanifold with weights equal to one away from branch points and branched along the two orbits $0 \times S^1$ and $S^1 \times 0$.

Remark 9.7. The support of a weighted branched submanifold is a rectifiable set in the sense of geometric measure theory [18]. Thus certain properties of weighted branched submanifolds follow from general properties of rectifiable sets, notably the existence of tangent spaces at almost all points (Lemma 9.10). However, weighted branched submanifolds are much simpler objects and we give direct proofs without referring to geometric measure theory. We also point out that our definition below of an orientation of a weighted branched submanifold differs from an orientation of a rectifiable set in [18], and compact oriented weighted branched submanifolds are not rectifiable currents in the sense of geometric measure theory (because of the rational weights).

9.1. The branched tangent bundle

Consider the bundle of Grassmannians of linear subspaces $F \subset T_xB$ that contain the tangent space of the G-orbit of x and have dimension $d + \dim \mathrm{G}$. We denote this Grassmannian bundle by

$$\mathrm{Gr}_d(TB/\mathfrak{g}) := \{(x,F) \mid x \in B,\ F \in \mathrm{Gr}_d(T_xB/\mathfrak{g})\}.$$

Proposition 9.8. *Let $\lambda : B \to \mathbb{Q}$ be a weighted branched d-submanifold of B. Then there exists a unique weighted branched d-submanifold*

$$T\lambda : \mathrm{Gr}_d(TB/\mathfrak{g}) \to \mathbb{Q}$$

such that

$$T\lambda(x, F) = \sum_{T_x M_i = F} \lambda_i \tag{21}$$

for any system of local branches (M_i, λ_i) near x. The branched submanifold $T\lambda$ of $\mathrm{Gr}_d(TB/\mathfrak{g})$ is called the tangent bundle *of λ.*

Proof. The proof has three steps.

Step 1: *If (M_i, λ_i), $i = 1, \ldots, m$, is a system of local branches of λ near x such that $x \in M_i$ for every i. Then $\xi^* x \in T_x M_i$ for every i and every $\xi \in \mathfrak{g}$.*

By assumption, $\lambda(x) = \sum_{i=1}^m \lambda_i$. Suppose, by contradiction that there exist an index j and an element $\xi \in \mathfrak{g}$ such that $\xi^* x \notin T_x M_j$. Then $\exp(t\xi)^* x \notin M_j$ for small positive t and hence $\lambda(\exp(t\xi)^* x) < \lambda(x)$, in contradiction to the equivariance axiom for branched submanifolds.

Step 2: *There exists a unique function $T\lambda : \mathrm{Gr}_d(TB/\mathfrak{g}) \to \mathbb{Q}$ that satisfies* (21) *for every system of local branches (M_i, λ_i) near x.*

The function $T\lambda$ is obviously uniquely determined by condition (21). We must prove that it is well defined. Let (M_i, λ_i), $i = 1, \ldots, m$, and (N_j, μ_j), $j = 1, \ldots, n$, be two systems of local branches in a common open neighbourhood U of x_0 such that

$$x_0 \in \bigcap_{i=1}^m M_i \cap \bigcap_{j=1}^n N_j.$$

We claim that there exist

- a positive integer ℓ,
- sequences $x_{k,\nu} \in M \setminus M^{\mathrm{b}}$ for $k = 1, \ldots, \ell$ such that $\lim_{\nu \to \infty} x_{k,\nu} = x_0$ for every k, and
- decompositions

$$\{1, \ldots, m\} = I_1 \cup \cdots \cup I_\ell, \quad \{1, \ldots, n\} = J_1 \cup \cdots \cup J_\ell,$$

  such that $I_k = \{i \mid x_{k,\nu} \in M_i\}$ and $J_k = \{j \mid x_{k,\nu} \in N_j\}$ for every k and every ν.

To see this note that, by Remark 9.3, there exists a sequence $x_{1,\nu} \in M_1 \setminus M^{\mathrm{b}}$ converging to x_0. Let $I_{1,\nu} \subset \{1, \ldots, m\}$ be the set of indices i such that $x_{1,\nu} \in M_i$ and, similarly, $J_{1,\nu} \subset \{1, \ldots, n\}$ be the set of indices j such that $x_{1,\nu} \in N_j$. Passing to a subsequence, if necessary, we may assume that the index sets $I_{1,\nu} =: I_1$ and $J_{1,\nu} =: J_1$ are independent of ν. If $I_1 = \{1, \ldots, m\}$ then $\lambda(x_{1,\nu}) = \lambda(x_0)$ for every ν and so $J_1 = \{1, \ldots, n\}$. Otherwise choose a sequence $x_{2,\nu} \in M \setminus \bigcup_{i \in I_1} M_i$ converging to x_0. Since $M \setminus M^{\mathrm{b}}$ is dense in M (see Remark 9.3), we may assume without loss of generality that $x_{2,\nu} \notin M^{\mathrm{b}}$. Now continue by induction to obtain the required sequences $x_{k,\nu}$, $k = 1, \ldots, \ell$.

With the existence of the sequences $x_{k,v}$ established we have

$$F_{k,v} := T_{x_{k,v}} M_i = T_{x_{k,v}} N_j$$

for every $i \in I_k$ and every $j \in J_k$, because $x_{k,v}$ is not a branch point of M. Moreover, by construction, the numbers

$$v_k := \lambda(x_{k,v}) = \sum_{i \in I_k} \lambda_i = \sum_{j \in J_k} \mu_j$$

are independent of v. It follows that

$$F_k := \lim_{v \to \infty} F_{k,v} = T_{x_0} M_i = T_{x_0} N_j$$

for every $i \in I_k$ and every $j \in J_k$. Hence

$$\sum_{T_x M_i = F} \lambda_i = \sum_{F_k = F} v_k = \sum_{T_x N_j = F} \mu_j.$$

This proves that the sum in (21) is independent of the choice of the local branches.

Step 3: *The function* $T\lambda$: $\mathrm{Gr}_d(TB/\mathfrak{g}) \to \mathbb{Q}$ *of Step* 2 *is a weighted branched submanifold of* $\mathrm{Gr}_d(TB/\mathfrak{g})$.

Equivariance follows from the fact that, if the weighted submanifolds (M_i, λ_i) are local branches of λ in U, then the weighted submanifolds $(g^* M_i, \lambda_i)$ are local branches of λ in $g^* U$. The function $T\lambda$ evidently satisfies the local structure axiom with local branches $TM_i := \{(x, T_x M_i) \mid x \in M_i\} \subset \mathrm{Gr}_d(TB/\mathfrak{g})$ in $\pi^{-1}(U) \subset \mathrm{Gr}_d(TB/\mathfrak{g})$ and weights λ_i. □

Definition 9.9. Let $\lambda : B \to \mathbb{Q}$ be a weighted branched d-submanifold of B with support M. A point $x \in M$ is called *singular* if

$$\#\{F \in \mathrm{Gr}_d(T_x B/\mathfrak{g}) \mid T\lambda(x, F) \neq 0\} > 1.$$

The set of singular points will be denoted by M^{s}.

Note that

$$M^{\mathrm{s}} \subset M^{\mathrm{b}}$$

for every weighted branched d-submanifold. In general, the set M^{b} can be considerably larger than M^{s}, although both sets are nowhere dense. Example 9.4 shows that the set $M \setminus M^{\mathrm{b}}$ can have arbitrarily small measure. In contrast, the next lemma shows that the set M^{s} always has measure zero.

Lemma 9.10. *Let λ be a weighted branched d-submanifold of B with support M and local branches $M_1, \ldots, M_m$ near x_0. Then, for every j, the set $M_j \cap M^{\mathrm{s}}$ has measure zero in M_j.*

Proof. Fix a number $j \in \{1, \ldots, m\}$ and, for $j' \neq j$, consider the set

$$C_{j'} := \{x \in M_{j'} \cap M_j \mid T_x M_{j'} \neq T_x M_j\},$$

where $T_x M_{j'}$ and $T_x M_j$ are understood as nonoriented subspaces of $T_x B$. Then each set $C_{j'}$ is a countable union of compact sets, namely of the sets $C_{j',\varepsilon}$ of all points $x \in M_{j'} \cap M_j$ such that $T_x M_{j'}$

contains a unit vector whose angle to T_xM_j is at least ε and whose open ε-neighbourhood is contained in U. Moreover,

$$M_j \cap M^{\mathrm{s}} = \bigcup_{j' \neq j} C_{j'}.$$

Now fix a number $j' \in \{1, \dots, m\} \setminus \{j\}$. Let $x \in C_{j'}$. Then there exists a neighbourhood $V \subset B$ of x such that the intersection $M_j \cap M_{j'} \cap V$ is contained in a codimension-1 submanifold of M_j. Hence the set $C_{j'} \cap V$ is contained in a codimension-1 submanifold of M_j. Since $C_{j'}$ is a countable union of compact sets it follows that the $C_{j'}$ can be covered by countably many codimension-1 submanifolds of M_j. Since this holds for every $j' \neq j$, it follows that $M_j \cap M^{\mathrm{s}}$ has measure zero. □

9.2. Orientations

Next we shall introduce the notion of an orientation of a branched submanifold. Consider the bundle of Grassmannians of oriented linear subspaces of T_xB that contain the tangent space of the G-orbit of x and have dimension $d + \dim \mathrm{G}$. We denote this Grassmannian bundle by

$$\mathrm{Gr}_d^+(TB/\mathfrak{g}) := \{(x, F) \mid x \in B,\ F \in \mathrm{Gr}_d^+(T_xB/\mathfrak{g})\}.$$

We write $-F$ for the subspace F equipped with the opposite orientation.

Definition 9.11. Let B be a finite dimensional manifold and G be a compact oriented Lie group which acts on B with finite isotropy. Let $\lambda : B \to \mathbb{Q}$ be a weighted branched d-submanifold of B. An *orientation* of λ is a function

$$\mu : \mathrm{Gr}_d^+(TB/\mathfrak{g}) \to \mathbb{Q}$$

with the following properties.

(**Equivariance**) $\mu(g^*x, g^*F) = \mu(x, F)$ for all $x \in B$, $F \in \mathrm{Gr}_d^+(T_xB/\mathfrak{g})$, and $g \in G$.

(**Local structure**) For each $x_0 \in B$ there exists a system of *oriented* local branches (M_i, λ_i), $i = 1, \dots, m$, in a neighbourhood U such that

$$\mu(x, F) = \sum_{T_xM_i = F} \lambda_i - \sum_{T_xM_i = -F} \lambda_i$$

for every $x \in U$.

Remark 9.12. Every orientation μ of λ satisfies

$$\mu(x, -F) = -\mu(x, F). \tag{22}$$

Note that μ can vanish on the Grassmannian $\mathrm{Gr}_d^+(T_xB/\mathfrak{g})$ for a point $x \in M$ when the oriented weights of the branches cancel each other out at x.

Remark 9.13. Every weighted branched d-submanifold $\lambda : B \to \mathbb{Q}$ admits an orientation. To see this, choose local oriented branches (M_i, λ_i) such that every branch appears twice, once with each orientation. Then the function $\mu \equiv 0$ satisfies the requirements of Definition 9.11.

Remark 9.14. If $\lambda : B \to \mathbb{Q}$ is the characteristic function of an ordinary submanifold $M \subset B$ then the oriented Grassmannian $\mathrm{Gr}_d^+(TM/\mathfrak{g})$ is a 2–1 covering over M. If M is orientable then an orientation corresponds to a continuous function μ: $\mathrm{Gr}_d^+(TM/\mathfrak{g}) \to \mathbb{Q} \cap [-1, 1]$ which satisfies (22). To see this, fix an orientation of M in the usual sense, let $x_0 \in M$ and denote $\mu_0 := \mu(x_0, T_{x_0}M)$. Choose a positively oriented local branch with weight $\lambda_1 := (1 + \mu_0)/2$ and a negatively oriented local branch with weight $\lambda_2 := (1 - \mu_0)/2$. If μ takes only values ± 1 it is equivalent to an orientation in the usual sense. If M is connected and not orientable then $\mu \equiv 0$ is the only orientation of λ.

Remark 9.15. In the case $d = 0$ the set $\mathrm{Gr}_0^+(T_xB/\mathfrak{g})$ is canonically isomorphic to $\{\pm \mathfrak{g}^* x\}$. In this case an orientation determines a function $B \to \mathbb{Q} : x \mapsto \mu(x, \mathfrak{g}^* x)$. We emphasize that the *contravariant* action determines the orientation and this is important when the dimension of G is odd.

Example 9.16. Consider a branched 1-submanifold λ of the plane $B = \mathbb{R}^2$ whose support M is the union of a circle and the graph of a smooth nonnegative function on the circle which vanishes on a closed interval $M_0 \subset S^1$ and is positive on the complement $S^1 \setminus M_0$. Assume that $\lambda(x) = 2$ for $x \in M_0$ and $\lambda(x) = 1$ for $x \in M \setminus M_0$. Then λ admits four orientations which are equal to ± 1 on $M \setminus M_0$. Two of these orientations vanish on M_0.

Remark 9.17. Definition 9.11 is more general than the definition of an oriented branched submanifold in [23]. In [23] it is required that the orientations of the local branches can be chosen such that they agree over the complement of the set M^{b} of the branch points. The orientation $\mu_{S,\sigma}$ of $\lambda_{S,\sigma}$ in Proposition 10.5 satisfies this condition. However, it is not necessary to impose this in order to obtain a well-defined notion of an integral over a compact oriented branched d-submanifold.

Example 9.18 (Product). The product of two weighted branched submanifolds $\lambda_i : B_i \to \mathbb{Q}$ is the weighted branched submanifold $\lambda : B_0 \times B_1 \to \mathbb{Q}$ defined by

$$\lambda(x_0, x_1) := \lambda_0(x_0)\lambda_1(x_1).$$

Orientations μ_i: $\mathrm{Gr}_{d_i}^+(TB_i/\mathfrak{g}_i) \to \mathbb{Q}$ of the λ_i induce an orientation

$$\mu: \mathrm{Gr}_{d_0+d_1}^+(T(B_0 \times B_1)/(\mathfrak{g}_0 \times \mathfrak{g}_1)) \to \mathbb{Q}$$

of λ via

$$\mu((x_0, x_1), F_0 \times F_1) := \mu_0(x_0, F_0)\mu_1(x_1, F_1).$$

9.3. Branched cobordisms

Compact weighted branched d-submanifolds of B form a (small) category. The morphisms are branched cobordisms. This requires the notion of a branched d-submanifolds with boundary. More precisely, let B be a smooth finite dimensional G-manifold with (G-invariant) boundary ∂B. A *weighted branched d-submanifold with boundary* $\lambda : B \to \mathbb{Q}$ is defined as in Definition 9.1 except that the local branches M_i are now submanifolds with boundary $\partial M_i = M_i \cap \partial B$ and they are required to be transverse to the boundary ∂B. The *boundary of* λ is defined as the restriction $\partial\lambda := \lambda|_{\partial B}$. If

$\mu\colon \mathrm{Gr}_d^+(TB/\mathfrak{g}) \to \mathbb{Q}$ is an orientation of λ, then the *boundary orientation* of $\partial\lambda$ is the function $\partial\mu\colon \mathrm{Gr}_d^+(T\partial B/\mathfrak{g}) \to \mathbb{Q}$ defined by

$$\partial\mu(x,\partial F) := \sum_v \mu(x, v\mathbb{R} \oplus \partial F)$$

for $x \in \partial B$ and $\partial F \in \mathrm{Gr}_{d-1}^+(T_x\partial B/\mathfrak{g})$, where the sum runs over all outward pointing unit vectors v.

Definition 9.19. Let B be a smooth finite dimensional G-manifold.

(i) Two compact weighted branched d-submanifolds $\lambda_0, \lambda_1 : B \to \mathbb{Q}$ are called *cobordant* if there exist a compact weighted branched $(d+1)$-submanifold $\lambda : [0,1] \times B \to \mathbb{Q}$ and a constant $\varepsilon > 0$ such that

$$\lambda_0(x) = \lambda(t,x), \quad \lambda_1(x) = \lambda(1-t,x)$$

for every $x \in B$ and every $t \in [0,\varepsilon]$. In this case λ is called a *compact weighted branched cobordism* from λ_0 to λ_1.

(ii) Two compact oriented weighted branched d-submanifolds (λ_0,μ_0), (λ_1,μ_1) of B are called *oriented cobordant* if there exist a compact oriented weighted branched $(d+1)$-submanifold (λ,μ) of $[0,1] \times B$ such that λ is a compact weighted branched cobordism from λ_0 to λ_1 and

$$\mu_0(x,F) = \mu((0,x), \mathbb{R}(1,0) \times F), \quad \mu_1(x,F) = \mu((1,x), \mathbb{R}(1,0) \times F)$$

for every $x \in B$ and every $F \in \mathrm{Gr}_d^+(T_xB/\mathfrak{g})$. In this case (λ,μ) is called a *compact oriented weighted branched cobordism* from (λ_0,μ_0) to (λ_1,μ_1).

Let $\lambda : B \to \mathbb{Q}$ be a weighted branched d-submanifold of B and $\lambda' : B \to \mathbb{Q}$ be a weighted branched d'-submanifold. Then λ and λ' are called *transverse* if any two subspaces $F, F' \subset T_xB$ such that $T\lambda(x,F) > 0$ and $T\lambda(x,F') > 0$ intersect transversally. In this case the product

$$\lambda\lambda' : B \to \mathbb{Q},$$

is again a weighted branched submanifold, called the *intersection of* λ *and* λ'. An orientation of B and orientations $\mu\colon \mathrm{Gr}_d^+(TB/\mathfrak{g}) \to \mathbb{Q}$ and $\mu'\colon \mathrm{Gr}_{d'}^+(TB/\mathfrak{g}) \to \mathbb{Q}$ of λ and λ', respectively, induce an orientation

$$\mu\mu'\colon \mathrm{Gr}^+_{d+d'-\dim B+\dim \mathrm{G}}(TB/\mathfrak{g}) \to \mathbb{Q}$$

of $\lambda\lambda'$ via

$$\mu\mu'(x,H) := \sum_{H = F\cap F'} \mu(x,F)\mu_1(x,F') \tag{23}$$

for $H \in \mathrm{Gr}^+_{d+d'-\dim B+\dim \mathrm{G}}(T_xB/\mathfrak{g})$.

Proposition 9.20. *Let* $\lambda' : B \to \mathbb{Q}$ *be a weighted branched* d'*-submanifold of* B *with closed support. Then the following holds*:

(i) *Every compact* (*oriented*) *weighted branched* d*-submanifold* $\lambda : B \to \mathbb{Q}$ *is* (*oriented*) *cobordant to a compact* (*oriented*) *weighted branched* d*-submanifold of* B *that is transverse to* λ'.

(ii) *If* $\lambda_0, \lambda_1 : B \to \mathbb{Q}$ *are* (*oriented*) *weighted branched d-submanifolds of B that are* (*oriented*) *cobordant and transverse to* λ', *then there exists a compact* (*oriented*) *weighted branched cobordism* $\lambda : [0,1] \times B \to \mathbb{Q}$ *from* λ_0 *to* λ_1 *such that* λ *is transverse* $[0,1] \times \lambda'$.

Proof. The transversality theory in [1] can be adapted to branched submanifolds as follows. A *multivalued vector field* on B is a weighted branched d-submanifold $\eta : TB \to \mathbb{Q}$ such that the branches of η are local vector fields on B and

$$\sum_{v \in T_x B} \eta(x,v) = 1$$

for every $x \in B$ (see Definition 10.1). The convolution of two such vector fields is defined by

$$\eta_0 * \eta_1(x,v) := \sum_{v_0 + v_1 = v} \eta_0(x, v_0)\eta_1(x, v_1).$$

Using cutoff functions one can show that, for every $x \in B$ and every $v \in T_x B$, there exists a multivalued vector field $\eta : TB \to \mathbb{Q}$ such that $\eta(x,v) \neq 0$. Hence, by using convolutions, one can construct a finite sequence of multivalued vector fields $\eta_1, \ldots, \eta_N : TB \to \mathbb{Q}$ along λ such that, for every $x \in B$ such that $\lambda(x) > 0$, there exists a spanning sequence $v_1, \ldots, v_N \in T_x B$ such that $\eta_i(x, v_i) > 0$. Now choose any G-invariant metric on B and, for $\varepsilon > 0$ sufficiently small, consider the function

$$\Lambda : \{\zeta \in \mathbb{R}^N \mid |\zeta| < \varepsilon\} \times B \to \mathbb{Q}$$

defined by

$$\Lambda(\zeta, x) := \frac{1}{N} \sum_{i=1}^{N} \sum_{x_i \in B} \lambda(x_i) \sum_{\substack{v_i \in T_{x_i} B \\ \exp_{x_i}(\zeta^i v_i) = x}} \eta_i(x_i, v_i)$$

for $\zeta = (\zeta^1, \ldots, \zeta^N) \in \mathbb{R}^N$ such that $|\zeta| < \varepsilon$ and $x \in B$. Then Λ is a weighted branched $(d+N)$-submanifold of $\mathbb{R}^N \times B$,

$$\Lambda(0,x) = \lambda(x),$$

and Λ is transverse to $\Lambda' := \mathbb{R}^N \times \lambda'$. Hence the intersection $\Lambda\Lambda'$ is a branched submanifold of $\mathbb{R}^N \times B$. Let $\zeta_1 \in \mathbb{R}^N$ be a sufficiently small common regular value of the projections from the branches of $\Lambda\Lambda'$ to $\mathbb{R}^N$. Then the compact branched submanifold

$$B \to \mathbb{Q} : x \mapsto \Lambda(\zeta_1, x)$$

is cobordant to λ and transverse to λ'. This proves (i). The proof of (ii) is similar. □

9.4. Integration

Let $\lambda : B \to \mathbb{Q}$ be a compact weighted branched d-submanifold of B with support M and let $\mu : \mathrm{Gr}_d^+(TB/\mathfrak{g}) \to \mathbb{Q}$ be an orientation of λ. We now explain how to integrate an equivariant differential form $\alpha \in \Omega_G^d(B)$ over (λ, μ). Abusing notation, we shall not indicate the dependence on μ in the notation. The integral is defined by

$$\int_{\lambda/\mathrm{G}} \alpha := \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{\lambda_{ij}}{|\mathrm{G}_i|} \int_{M_{ij}\cap\phi_i(U_i)} \rho_i \alpha_A, \tag{24}$$

where $A \in \Omega^1(B, \mathfrak{g})$ is a connection 1-form on B, $(U_i, \phi_i, \mathrm{G}_i)$, $i = 1, \ldots, N$, are local slices of the G-action on B such that the sets $\mathrm{G}^*\phi_i(U_i)$ cover M, the pairs (M_{ij}, λ_{ij}), $j = 1, \ldots, m_i$, are the oriented weighted branches of M in a neighbourhood of $\phi_i(U_i)$, and the functions $\rho_i : B \to [0, 1]$ form a G-invariant partition of unity over M such that $\operatorname{supp} \rho_i \subset \mathrm{G}^*\phi_i(U_i)$.

Proposition 9.21. (i) *Integral* (24) *is independent of the oriented local branches, the connection, the local slices, and the partition of unity used to define it.*

(ii) *If* $\beta \in \Omega_{\mathrm{G}}^{d-1}(B)$ *and* $\lambda : B \to \mathbb{Q}$ *is a compact oriented weighted branched* d*-submanifold with boundary then*

$$\int_{\lambda/\mathrm{G}} d_{\mathrm{G}}\beta = \int_{\partial\lambda/\mathrm{G}} \beta.$$

Proof. Fix a local slice $(U_0, \phi_0, \mathrm{G}_0)$. Suppose that (M_i, λ_i), $i = 1, \ldots, m$, and (N_j, μ_j), $j = 1, \ldots, n$, are two collections of oriented local branches in a neighbourhood of $\phi_0(U_0)$, such that the orientations of both collections of local branches are compatible with μ as in Definition 9.11. Suppose that $\alpha \in \Omega_{\mathrm{G}}^d(B)$ is supported in $\mathrm{G}^*\phi_0(U_0)$ and let $A \in \Omega^1(B, \mathfrak{g})$ be a connection 1-form. We must prove that

$$\sum_{i=1}^{m} \lambda_i \int_{M_i\cap\phi_0(U_0)} \alpha_A = \sum_{j=1}^{n} \mu_j \int_{N_j\cap\phi_0(U_0)} \alpha_A. \tag{25}$$

To see this recall from Lemma 9.10 that each set $M_i \cap M^{\mathrm{s}}$ and $N_j \cap M^{\mathrm{s}}$ has measure zero. Moreover, by Definition 9.9, the projection from the support of $T\lambda$ to B is bijective over $M \setminus M^{\mathrm{s}}$. Hence the tangent spaces of the submanifolds $M_i \setminus M^{\mathrm{s}}$ and $N_j \setminus M^{\mathrm{s}}$ agree at each intersection point. Now choose a finite collection of G-invariant disjoint Borel sets $Q_1, \ldots, Q_\ell \subset M \setminus M^{\mathrm{s}}$ such that

$$M \cap \phi_0(U_0) \setminus M^{\mathrm{s}} = Q_1 \cup \cdots \cup Q_\ell,$$

$$M_i \cap Q_k \neq \emptyset \;\Rightarrow\; Q_k \subset M_i,$$

$$N_j \cap Q_k \neq \emptyset \;\Rightarrow\; Q_k \subset N_j$$

for all i, j, and k. Define the measurable functions $f_k : M \cap \phi_0(U_0) \to [0, 1]$ by

$$f_k(x) := \begin{cases} 1 & \text{if } x \in Q_k, \\ 0 & \text{if } x \notin Q_k. \end{cases}$$

Moreover, choose finite sequences $i_k \in \{1, \ldots, m\}$ and $j_k \in \{1, \ldots, n\}$ such that $Q_k \subset M_{i_k} \cap N_{j_k}$ for all k. Then, by Definition 9.11,

$$\sum_{i=1}^{m} \lambda_i \int_{M_i\cap\phi_0(U_0)} \alpha_A = \sum_{k=1}^{\ell} \sum_{i=1}^{m} \lambda_i \int_{M_i\cap\phi_0(U_0)\setminus M^{\mathrm{s}}} f_k \alpha_A$$

$$= \sum_{k=1}^{\ell} \int_{M_{i_k} \cap \phi_0(U_0) \setminus M^s} \mu(x, T_x M_{i_k}) f_k \alpha_A$$

$$= \sum_{k=1}^{\ell} \int_{N_{j_k} \cap \phi_0(U_0) \setminus M^s} \mu(x, T_x N_{j_k}) f_k \alpha_A$$

$$= \sum_{k=1}^{\ell} \sum_{j=1}^{n} \mu_j \int_{N_j \cap \phi_0(U_0) \setminus M^s} f_k \alpha_A$$

$$= \sum_{j=1}^{n} \mu_j \int_{N_j \cap \phi_0(U_0)} \alpha_A.$$

This proves (25). It follows that the integral (24) is independent of the choice of the local branches and the partition of unity used to define it. To prove (ii), suppose first that β is supported in an open set $\mathrm{G}^* \phi_i(U_i)$ and choose a partition of unity such that ρ_i is equal to one on the support of β. Then the result follows from Stokes' theorem and the fact that $(d_G \beta)_A = d\beta_A$ (Theorem 3.8(ii)). To prove (ii) in general, consider the form $\sum_i d_{\mathrm{G}}(\rho_i \beta)$ for a suitable G-invariant partition of unity ρ_i. The independence of the connection A now follows from (ii) and Theorem 3.8(iii). The independence of the local slices follows as in the proof of Proposition 4.2. □

9.5. Intersection numbers

Suppose that B is oriented and (λ, μ) and (λ', μ') are compact oriented weighted branched submanifolds of B that intersect transversally. If their dimensions satisfy

$$d + d' = \dim B - \dim \mathrm{G},$$

then the intersection $(\lambda\lambda', \mu\mu')$ is a compact oriented weighted branched 0-submanifold. This is just a collection of finitely many G-orbits $[x]$ with isotropy subgroups G_x and orientations $\mu(x, \mathfrak{g}^* x) \in \mathbb{Q}$. In this case the *intersection number* of λ and λ' is defined by

$$\lambda \cdot \lambda' := \int_{\lambda\lambda'/\mathrm{G}} 1 = \sum_{[x]} \sum_{\mathfrak{g}^* x = F \cap F'} \frac{\mu(x, F)\mu'(x, F')}{|\mathrm{G}_x|}.$$

Here the first sum runs over all G-orbits $[x]$ in B and the second sum over all pairs $(F, F') \in \mathrm{Gr}_d^+(T_x B/\mathfrak{g}) \times \mathrm{Gr}_{d'}^+(T_x B/\mathfrak{g})$.

Proposition 9.22. *The intersection number depends only on the oriented cobordism classes of λ and λ'.*

Proof. Suppose that λ_0 is oriented cobordant to λ_1 and that λ_0 and λ_1 are transverse to λ'. Then, by Proposition 9.20, there exists a compact oriented weighted branched cobordism λ from λ_0 to λ_1 that is transverse to $[0,1] \times \lambda'$. Hence the intersection $\lambda([0,1] \times \lambda')$ is a (1-dimensional) compact oriented weighted branched cobordism from $\lambda_0\lambda'$ to $\lambda_1\lambda'$. Hence it follows from Proposition 9.21 that $\lambda_0 \cdot \lambda' = \lambda_1 \cdot \lambda'$. $\square$

Now consider the case $G = \{\mathbb{1}\}$. Let X be a smooth compact oriented finite dimensional manifold with boundary ∂X and (λ, μ) be a compact oriented weighted branched d-submanifold of X whose support M does not intersect the boundary ∂X. Let Y be a compact oriented smooth manifold with boundary such that

$$d + \dim Y = \dim X.$$

A smooth map $f : (Y, \partial Y) \to (X, \partial X)$ is called *transverse to* λ if the graphs of f and $Y \times \lambda$ are transverse as weighted branched manifolds of $Y \times X$, or equivalently, if f is transverse to every branch of λ. If this holds then it follows from the definition of a branched submanifold that $f^{-1}(M) \subset Y$ is a finite set. The intersection number of f with (λ, μ) is given by

$$f \cdot \lambda = \sum_{y \in f^{-1}(M)} \sum_{j=1}^{m_y} \lambda_i \varepsilon(y; f, M_i),$$

where $U_y \subset X$ is an open neighbourhood of $f(y)$, the pairs (M_i, λ_i) for $i = 1, \ldots, m_y$ are the local oriented weighted branches of (λ, μ) in U_y, and the intersection number $\varepsilon(y; f, M_i)$ is defined to be ± 1 according to whether or not the orientations agree in the decomposition

$$T_{f(y)}X = \operatorname{im} \mathrm{d}f(y) \oplus T_{f(y)}M_i.$$

Applying Proposition 9.22 with $G = \{\mathbb{1}\}$ to the graph of f and the branched submanifold $Y \times \lambda$ of $Y \times X$, we find that the intersection number depends only on the homotopy class of f and the oriented cobordism class of (λ, μ).

9.6. Rational cycles

The next theorem asserts that, in the case $G = \{\mathbb{1}\}$, every compact oriented weighted branched submanifold determines a rational homology class and that the intersection corresponds to the intersection product in homology.

Theorem 9.23. *Let Z be a smooth finite dimensional manifold and $\lambda : Z \to \mathbb{Q}$ be a compact oriented weighted branched d-submanifold of Z.*

(i) *There exists a unique rational homology class $[\lambda] \in H_d(Z; \mathbb{Q})$ in singular homology such that*

$$\langle [\alpha], [\lambda] \rangle = \int_\lambda \alpha$$

for every closed d-form $\alpha \in \Omega^d(Z)$.

(ii) *The homology class $[\lambda]$ depends only on the oriented cobordism class of λ.*

(iii) *If Z is oriented and $\lambda' : Z \to \mathbb{Q}$ is a compact oriented weighted branched submanifold of Z that intersects λ transversally, then*

$$[\lambda\lambda'] = [\lambda] \cdot [\lambda'],$$

where $\cdot$ denotes the intersection pairing on singular homology.

Proof. The proof has eight steps.

Step 1: *We may assume without loss of generality that Z is oriented.*

Let $\pi:\tilde{Z}\to Z$ be the oriented double cover and denote by $\tilde{\lambda}:\tilde{Z}\to\mathbb{Q}$ the composition of λ with π. Assuming assertion (i) in the oriented case we obtain a homology class $[\tilde{\lambda}]\in H_d(\tilde{Z};\mathbb{Q})$. The required homology class on Z is then given by $2[\lambda]:=\pi_*[\tilde{\lambda}]\in H_d(Z;\mathbb{Q})$.

Step 2: *Let $X\subset Z$ be a compact neighbourhood of the support M of λ with smooth boundary ∂X. Let $\alpha\in\Omega^d(X)$ be a closed differential form whose cohomology class $[\alpha]\in H^d(X;\mathbb{R})$ is dual to a smooth map $f:(Y,\partial Y)\to(X,\partial X)$. Then*

$$\int_\lambda \alpha = f\cdot\lambda. \tag{26}$$

To see this note that, by a standard general position argument, f can be chosen transverse to λ. Suppose first that f is an embedding. Then there exists a closed d-form $\alpha_f\in\Omega^d(X)$ such that $\alpha-\alpha_f$ is exact, α_f is supported in a small tubular neighbourhood of $f(Y)$, and the pullback of α_f to the normal bundle of $f(Y)$ is a Thom form. Hence, by Proposition 9.21(ii),

$$\int_\lambda \alpha = \int_\lambda \alpha_f.$$

Now formula (26), with $k\alpha$ replaced by α_f, follows from the fact that the integral of α_f over a local branch M_i of λ is localized near the intersection point $f(y)\in M_i$ and is equal to the intersection number $\varepsilon(y;f,M_i)$ at this point.

The nonembedded case can be reduced to the embedded case by replacing f by the graph of f and α by a closed n-form $\tau_f\in\Omega^n(Y\times X)$ such that τ_f is supported in a tubular neighbourhood $U_f\subset Y\times X$ of the graph of f. Then $\alpha-\int_Y\tau_f\in\Omega^d(X)$ is exact, where $\int_Y$ denotes integration over the fibre. Hence

$$\int_\lambda \alpha = \int_{Y\times\lambda}\tau_f = \mathrm{graph}(f)\cdot(Y\times\lambda) = f\cdot\lambda.$$

Here $Y\times\lambda$ denotes the induced branched n-submanifold of $Y\times X$ with support $Y\times M$ and the orientation $Y\times\mu$ on $Y\times\lambda$ is induced by the orientation of Y and μ. In the above equation the first equality follows from Proposition 9.21(ii), the second from the embedded case, and the last from the definition of the intersection number. This proves (26).

Step 3: *If $\alpha\in\Omega^d(Z)$ represents a rational cohomology class $[\alpha]\in H^d(Z;\mathbb{Q})$ then $\int_\lambda\alpha\in\mathbb{Q}$.*

Let $X\subset Z$ be a compact neighbourhood of the support M of λ with smooth boundary ∂X and denote by $\iota:X\to Z$ the obvious inclusion. Then $\iota^*\alpha$ represents a singular cohomology class $[\iota^*\alpha]\in H^d(X;\mathbb{Q})$. The Poincaré dual of $[\iota^*\alpha]$ is a relative rational homology class

$$\mathrm{PD}([\iota^*\alpha])\in H_{n-d}(X,\partial X;\mathbb{Q}),\quad n:=\dim X.$$

Now for every such class there exist an integer k, a compact oriented smooth $(n-d)$-manifold Y with boundary, and a smooth map $f:(Y,\partial Y)\to(X,\partial X)$ such that the image of $[Y]\in H_{n-d}(Y,\partial Y;\mathbb{Q})$ under f_* is equal to

$$f_*[Y] = k\mathrm{PD}([\iota^*\alpha])\in H^*(X,\partial X;\mathbb{Q})$$

(see [10, Corollary 27.13]). Here we denote by $[Y]$ the image of the fundamental class (understood as an integral homology class) under the homomorphism $H_*(Y,\partial Y;\mathbb{Z}) \to H_*(Y,\partial Y;\mathbb{Q})$. Hence, by Step 2,

$$k\int_\lambda \alpha = f\cdot\lambda \in \mathbb{Q}.$$

This proves Step 3.

Step 4: *We prove* (i) *and* (ii).

By de Rham's theorem, every rational singular cohomology class can be represented by a differential form $\alpha\in\Omega^d(Z)$ such that the integral of α over every smooth integral cycle is a rational number. By Step 3, $\int_\lambda \alpha\in\mathbb{Q}$ for every such differential form α. Thus integration over λ defines a homomorphism $H^d(X;\mathbb{Q})\to\mathbb{Q}$. Now the universal coefficient theorem asserts that

$$H_d(X;\mathbb{Q}) \cong \mathrm{Hom}(H^d(X;\mathbb{Q}),\mathbb{Q}).$$

Hence there exists a rational cycle in X (and hence in Z) such that integration over λ is equal to integration over this rational cycle. This proves (i). Assertion (ii) follows from Proposition 9.21.

Step 5: *Assume $d+d'=\dim Z$ and let $\tau_\lambda\in\Omega^{\dim Z-d}(Z)$ be a closed form with compact support that is dual to* $[\lambda]$. *Then*

$$\int_{\lambda'} \tau_\lambda = \lambda\cdot\lambda' \tag{27}$$

for every oriented weighted branched d'-submanifold $\lambda' : Z\to\mathbb{Q}$ that is transverse to λ and has closed support.

Choose a compact neighbourhood X of the support of λ with smooth boundary ∂X such that each branch of λ' intersects X in a closed submanifold and is transverse to the boundary. We may also choose X such that each of these branches intersects the support of λ in precisely one point. By (i) and Poincaré duality, there exists a closed form $\tau_\lambda\in\Omega^{\dim Z-d}(X)$ such that $\operatorname{supp}\tau_\lambda\subset X\setminus\partial X$ and

$$\int_\lambda \alpha = \int_X \alpha\wedge\tau_\lambda$$

for every closed form $\alpha\in\Omega^d(X)$. Denote by

$$M'_1,\ldots,M'_k\subset X$$

the intersections of the oriented branches of λ' with X and let $\lambda'_1,\ldots,\lambda'_k$ be the corresponding rational weights. For each j choose a differential form $\tau'_j\in\Omega^d(X)$ with support near M'_j such that τ'_j is a Thom form on the normal bundle of M'_j. Then, By Corollary 6.3,

$$\int_X \beta\wedge\tau'_j = \int_{M'_j}\beta$$

for every closed form $\beta\in\Omega^*(X)$ with $\operatorname{supp}\beta\subset X\setminus\partial X$. Hence

$$\int_{\lambda'}\tau_\lambda = \sum_{j=1}^k \lambda'_j\int_{M'_j}\tau_\lambda$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} \lambda'_j \int_X \tau_\lambda \wedge \tau'_j \\
&= (-1)^{dd'} \sum_{j=1}^{k} \lambda'_j \int_X \tau'_j \wedge \tau_\lambda \\
&= (-1)^{dd'} \sum_{j=1}^{k} \lambda'_j \int_\lambda \tau'_j \\
&= (-1)^{dd'} \sum_{j=1}^{k} \lambda'_j M'_j \cdot \lambda \\
&= (-1)^{dd'} \lambda' \cdot \lambda \\
&= \lambda \cdot \lambda'.
\end{aligned}
$$

Here the fifth equality follows from (26). Thus we have proved (27).

Step 6: *Let $\lambda' : Z \to \mathbb{Q}$ be an oriented weighted branched d'-submanifold of Z with closed support and $Y \subset Z$ be a smooth oriented submanifold that is transverse to λ' and closed as a subset of Z. Then*

$$\int_{\lambda'} \alpha \wedge \tau_Y = \int_{Y \cap \lambda'} \alpha \tag{28}$$

for every compactly supported closed form $\alpha \in \Omega^{d' - \operatorname{codim} Y}(X)$. Here $\tau_Y \in \Omega^{\operatorname{codim} Y}(Z)$ is a Thom form for the normal bundle of Y.

The branched $(d' - \operatorname{codim} Y)$-submanifold $Y \cap \lambda'$ is defined by

$$(Y \cap \lambda')(z) := \begin{cases} \lambda'(z) & \text{if } z \in Y, \\ 0 & \text{if } z \in Z \setminus Y. \end{cases}$$

The orientation of $Y \cap \lambda'$ is defined by (23) with μ given by the orientation of Y. Suppose first that $W \subset Z$ is a compact oriented submanifold which is transverse to Y, λ', and $Y \cap \lambda'$, and that $\alpha = \tau_W$ is a Thom form for the normal bundle of W. Then

$$\int_{\lambda'} \tau_W \wedge \tau_Y = (W \cap Y) \cdot \lambda' = W \cdot (Y \cap \lambda') = \int_{Y \cap \lambda'} \tau_W.$$

Here the first and last equalities follow from Step 5. This proves Step 6 in the case $\alpha = \tau_W$. The general case can be reduced to the case $\alpha = \tau_W$ as in the proof of Step 2.

Step 7: *Assume $d + d' > \dim Z$ and let $\tau_\lambda \in \Omega^{\dim Z - d}(Z)$ be a closed form with compact support that is dual to $[\lambda]$. Then*

$$\int_{\lambda'} \alpha \wedge \tau_\lambda = \int_{\lambda\lambda'} \alpha \tag{29}$$

for every closed form $\alpha \in \Omega^{d + d' - \dim Z}(X)$ and every oriented weighted branched d'-submanifold $\lambda' : Z \to \mathbb{Q}$ that is transverse to λ and has closed support.

We assume first that $\alpha = \tau_Y$ is dual to a smooth submanifold $Y \subset X$ with boundary $\partial Y = Y \cap \partial X$ and that Y is transverse to λ, λ', and $\lambda\lambda'$. Then

$$\begin{aligned}\int_{\lambda'} \tau_Y \wedge \tau_\lambda &= (-1)^{\operatorname{codim} Y \cdot \operatorname{codim} \lambda} \int_{\lambda'} \tau_\lambda \wedge \tau_Y \\ &= (-1)^{\operatorname{codim} Y \cdot \operatorname{codim} \lambda} \int_{Y \cap \lambda'} \tau_\lambda \\ &= (-1)^{\operatorname{codim} Y \cdot \operatorname{codim} \lambda} \lambda \cdot (Y \cap \lambda') \\ &= Y \cdot (\lambda\lambda') \\ &= \int_{\lambda\lambda'} \tau_Y.\end{aligned}$$

Here the second equality follows from Step 6 and the third and last equalities follow from Step 5. This proves Step 7 in the case $\alpha = \tau_Y$. The general case can be reduced to the case $\alpha = \tau_Y$ as in the proof of Step 2.

Step 8: *We prove* (iii).

Let τ_λ, $\tau_{\lambda'}$, $\tau_{\lambda\lambda'}$ be closed forms on Z with compact support that are dual to $[\lambda]$, $[\lambda']$, $[\lambda\lambda']$, respectively. Then the homological intersection pairing $[\lambda] \cdot [\lambda']$ is, by definition, Poincaré dual to the cohomology class of $\tau_\lambda \wedge \tau_{\lambda'}$. Now, by Step 7,

$$\int_Z \alpha \wedge \tau_\lambda \wedge \tau_{\lambda'} = \int_{\lambda'} \alpha \wedge \tau_\lambda = \int_{\lambda\lambda'} \alpha = \int_Z \alpha \wedge \tau_{\lambda\lambda'}$$

for every closed form $\alpha \in \Omega^{d+d'-\dim Z}(Z)$. Hence, by de Rham's theorem, the forms $\tau_\lambda \wedge \tau_{\lambda'}$ and $\tau_{\lambda\lambda'}$ represent the same cohomology classes in the compactly supported real cohomology of Z. Hence, in $H_*(Z; \mathbb{R})$,

$$[\lambda] \cdot [\lambda'] = \mathrm{PD}([\tau_\lambda \wedge \tau_{\lambda'}]) = \mathrm{PD}([\tau_{\lambda\lambda'}]) = [\lambda\lambda'].$$

By the universal coefficient theorem, this continues to hold in $H_*(Z; \mathbb{Q})$. □

Remark 9.24. Let Z be a smooth finite dimensional manifold and let $a \in H_d(Z; \mathbb{Q})$ be a rational homology class. Then there exists a compact oriented weighted branched d-submanifold $\lambda : Z \to \mathbb{Q}$ such that $a = [\lambda]$. Indeed, Thom has shown in [24] that there exists a positive integer k and a compact oriented submanifold $M \subset Z$ such that $ka = [M]$. Now just take the weighted branched submanifold with support M and weight $1/k$.

Example 9.25. Let $\iota : \mathbb{C}P^2 \to S^m$ be an embedding. Then the characteristic function $\lambda := \chi_{\iota(\mathbb{C}P^2)} : S^m \to \mathbb{Q}$ is a compact oriented weighted branched 4-submanifold of S^m which is homologous to zero but is not compact oriented weighted branched cobordant to the empty submanifold. The proof requires a refinement of the notion of an integral over a branched submanifold and a stronger notion of singular points. Namely one can introduce the set $M^{s,\infty}$ of all points x in the support of λ such that there are two local branches M_j and $M_{j'}$ passing through x which do not agree up to infinite order at x. Then one can deduce from Lemma 9.10 that the set $M_j \cap M^{s,\infty}$ has measure zero for every branch M_j. Now the notion of an integral can be extended to differential forms which are defined only on the support of λ and do not necessarily extend to the ambient space. The differential forms ω_j and $\omega_{j'}$ on two local branches M_j and $M_{j'}$ are required to agree on $M_j \cap M_{j'} \setminus M^{s,\infty}$. It then follows as in the proof of Proposition 9.21 that the integral is well defined and that Stokes' theorem continues to hold in this situation. This refined version of the integral can now be used to prove that the first Pontryagin number is well defined for a compact oriented weighted branched 4-submanifold and is an invariant of the compact oriented weighted branched cobordism class. Now the Hirzebruch signature theorem asserts that the first Pontryagin number of a smooth 4-manifold is equal to three times the signature, and hence is nonzero in our example. Hence an embedded projective plane cannot be compact oriented weighted branched cobordant to the empty submanifold.

We close this section with a conjecture.

Conjecture 9.26. *For every compact oriented weighted branched d-submanifold $\lambda : Z \to \mathbb{Q}$ there exists a rational cycle in Z which represents the class $[\lambda]$ and takes values in the support of λ.*

Note that the conjecture follows from Theorem 9.23 if the support of λ is the retract of an open neighbourhood in Z. But Example 9.4 shows that this need not be the case.

10. Multivalued perturbations

In this section we show how weighted branched submanifolds arise as zero sets of multivalued sections. The main theorem asserts that the Euler class of a finite dimensional G-moduli problem can be defined by integration over such a zero set. This implies rationality of the Euler class.

10.1. Multivalued sections

Definition 10.1. Suppose that $\pi : E \to B$ is a finite dimensional fibre bundle and G is a compact Lie group that acts on E and B with finite isotropy such that the projection π is G-equivariant. A *multivalued section* of E is a weighted branched submanifold

$$\sigma : E \to \mathbb{Q} \cap [0, \infty)$$

with the following properties.

(**Equivariance**) $\sigma(g^*x, g^*e) = \sigma(x, e)$ for all $x \in B$, $e \in E_x$, and $g \in G$.

(**Local structure**) For each $x_0 \in B$ there exist an open neighbourhood U of x_0, finitely many smooth sections $s_1, \dots, s_m : U \to E$, and finitely many positive rational numbers $\sigma_1, \dots, \sigma_m$ such that, for every $x \in U$,

$$\sum \sigma_i = 1, \quad \sigma(x, e) = \sum_{s_i(x) = e} \sigma_i.$$

Two multivalued sections σ_0, σ_1 are called *transverse* if they are transverse as weighted branched submanifolds. They are called *homotopic* if there exists a multivalued section σ of the pullback bundle $[0, 1] \times E \to [0, 1] \times B$ such that $\sigma|_{\{0\}\times E} = \sigma_0$ and $\sigma|_{\{1\}\times E} = \sigma_1$.

Remark 10.2. If $\sigma : E \to \mathbb{Q}$ is a multivalued section then, for every $x \in B$, the set

$$\Sigma(x) := \{e \in E_x \mid \sigma(x,e) > 0\}$$

is finite and $\sum_{e\in\Sigma(x)} \sigma(x,e) = 1$. Moreover, $\Sigma(x) = \{s_1(x),\dots,s_m(x)\}$, where the s_j are the local branches of σ.

Example 10.3. Let X and Y be manifolds on which G acts with finite isotropy. Then a *multivalued map* from X to Y is a multivalued section $\phi : X \times Y \to \mathbb{Q}$ of the trivial bundle $X \times Y \to X$. Suppose that ϕ_i are multivalued maps from X_i to Y. They give rise to weighted branched submanifolds $\sigma_i : X_0 \times X_1 \times Y \to \mathbb{Q}$, given by

$$\sigma_0(x_0,x_1,y) := \phi_0(x_0,y), \quad \sigma_1(x_0,x_1,y) := \phi_1(x_1,y).$$

If $\dim X_0 + \dim X_1 = \dim Y + \dim \mathrm{G}$, X_0 and X_1 are compact, X_0, X_1, Y, and G are oriented, and G acts on all three manifolds by orientation preserving diffeomorphisms, then there is an intersection number $\phi_0 \cdot \phi_1 \in \mathbb{Q}$. Proposition 9.22 implies that this number depends only on the homotopy classes of ϕ_0 and ϕ_1 (through multivalued maps).

Proposition 10.4. *Let $\sigma : E \to \mathbb{Q}$ be a multivalued section of a* G*-quivariant fibre bundle $\pi : E \to B$. Then the following holds*:

(i) *σ induces a map $\sigma^* : \Omega^*_G(E) \to \Omega^*_G(B)$ which is locally given by*

$$\sigma^*\alpha = \sum_i \sigma_i s_i^*\alpha. \tag{30}$$

(ii) *The map σ^* commutes with the differential d_G*:

$$d_G \circ \sigma^* = \sigma^* \circ d_G : \Omega^*_G(E) \to \Omega^{*+1}_G(B).$$

(iii) *If two multivalued sections σ_0, σ_1 are homotopic then there exists a linear map $Q : \Omega^*_G(E) \to \Omega^{*-1}_G(B)$ such that*

$$\sigma_1^* - \sigma_0^* = d_G \circ Q + Q \circ d_G : \Omega^*_G(E) \to \Omega^*_G(B).$$

(iv) *For every equivariant differential form $\alpha \in \Omega^d_{\mathrm{G}}(E)$ and every compact oriented weighted branched d-submanifold $\lambda : B \to \mathbb{Q}$ we have*

$$\int_{\lambda\sigma/\mathrm{G}} \alpha = \int_{\lambda/\mathrm{G}} \sigma^*\alpha,$$

where $\lambda\sigma : E \to \mathbb{Q}$ is the compact oriented branched d-submanifold defined by $\lambda\sigma(x,e) := \lambda(x)\sigma(x,e)$.

Proof. Define σ^* by Eq. (30). To prove that this is well defined, let (s_i, σ_i) and (t_j, τ_j) be two systems of local sections near $x_0 \in B$. Since the set of regular points is open and dense, we only need to prove the equation

$$\sum_i \sigma_i (s_i^*\alpha)_x = \sum_j \tau_j (t_i^*\alpha)_x$$

at points x such that (x,e) is regular for all $e \in E_x$ with $\sigma(x,e) > 0$. At such a point, $\mathrm{d}s_i(x) = \mathrm{d}t_j(x)$ for all i, j such that $s_i(x) = t_j(x) = e$. Given $e \in E_x$ with $\sigma(x,e) > 0$, choose indices i_e and j_e such that $s_{i_e}(x) = t_{j_e}(x) = e$. Then

$$\begin{aligned}\sum_i \sigma_i (s_i^*\alpha)_x(v_1,\ldots,v_k) &= \sum_i \sigma_i \alpha_{s_i(x)}(\mathrm{d}s_i(x)v_1,\ldots,\mathrm{d}s_i(x)v_k) \\ &= \sum_{e\,:\,\sigma(x,e)>0}\ \sum_{i\,:\,s_i(x)=e} \sigma_i \alpha_{s_i(x)}(\mathrm{d}s_i(x)v_1,\ldots,\mathrm{d}s_i(x)v_k) \\ &= \sum_{e\,:\,\sigma(x,e)>0} \sigma(x,e)\alpha_{(x,e)}(\mathrm{d}s_{i_e}(x)v_1,\ldots,\mathrm{d}s_{i_e}(x)v_k) \\ &= \sum_{e\,:\,\sigma(x,e)>0} \sigma(x,e)\alpha_{(x,e)}(\mathrm{d}t_{j_e}(x)v_1,\ldots,\mathrm{d}t_{j_e}(x)v_k) \\ &= \sum_j \tau_j (t_j^*\alpha)_x(v_1,\ldots,v_k).\end{aligned}$$

A similar argument shows that σ^* is G-equivariant, i.e.

$$\sigma^* \circ g^* = g^* \circ \sigma^*$$

for $g \in G$. So σ^* maps G-equivariant forms to G-equivariant forms. This proves (i).

We prove (ii). By G-equivariance, we have

$$\sigma^* \circ \iota(Y_\xi)\alpha = \iota(X_\xi) \circ \sigma^*\alpha$$

for $\alpha \in \Omega^*(E)$ and $\xi \in \mathfrak{g}$, where $X_\xi \in \mathrm{Vect}(B)$ denotes the infinitesimal action on B and $Y_\xi \in \mathrm{Vect}(E)$ the infinitesimal action on E. Since σ^* also commutes with d, it commutes with d_G.

For the proof of (iii) we only sketch the argument. The local formula

$$X_t(x,e) = \sum_{i\,:\,s_{ti}(x)=e} \sigma_{ti} \frac{\mathrm{d}}{\mathrm{d}t} s_{ti}(x)$$

defines a G-invariant multivalued vector field along σ. The operator

$$Q : \Omega^*(E) \to \Omega^{*-1}(B), \quad Q\alpha := \int_0^1 \sigma_t^* \iota(X_t)\alpha\, \mathrm{d}t$$

is G-equivariant and satisfies $\iota(X_\xi) \circ Q + Q \circ \iota(X_\xi) = 0$ for $\xi \in \mathfrak{g}$. Thus Q maps G-equivariant forms to G-equivariant forms and

$$d_G \circ Q + Q \circ d_G = d \circ Q + Q \circ d = \sigma_1^* - \sigma_0^*.$$

This proves (iii). Assertion (iv) follows directly from the definitions. □

10.2. The zero set of a multivalued section

Now let (B, E, S) be a finite dimensional regular G-moduli problem. A multivalued section σ is transverse to S if and only if $S - s_i$ is transverse to the zero section for each s_i in the local structure axiom.

It is sometimes useful to think of a multivalued section σ as a function which assigns to each $x \in B$ the discrete probability measure $\sum_e \sigma(x,e)\delta_e$ on the fibre E_x. Convolution of measures gives rise to a convolution operation $(\sigma_0, \sigma_1) \mapsto \sigma_0 * \sigma_1$ on multivalued sections given by

$$\sigma_0 * \sigma_1(x,e) := \sum_{e_0 + e_1 = e} \sigma_0(x, e_0) + \sigma_1(x, e_1).$$

This operation is commutative and associative and has a neutral element given by $\sigma(x,0) = 1$ for all $x \in B$. There is no inverse and so convolution gives only a semigroup structure.

Pushforward of measures under dilations $(x,e) \mapsto (x, te)$ gives rise to a multiplication of multivalued sections by G-invariant functions $f : B \to \mathbb{R}$,

$$(f\sigma)(x,e) := \sum_{f(x)e' = e} \sigma(x, e').$$

Convolution is distributive over multiplication by functions.

Proposition 10.5. *Let (B, E, S) be a finite dimensional regular* G*-moduli problem of index $d = \operatorname{index}(S) = \dim B - \operatorname{rank} E - \dim \mathrm{G}$ and $\sigma : E \to \mathbb{Q}$ be a multivalued section that is transverse to S. Then the function $\lambda_{S,\sigma} : B \to \mathbb{Q}$ defined by*

$$\lambda_{S,\sigma}(x) := \sigma(x, S(x))$$

is a weighted branched d-submanifold of B. Moreover, there exists a unique orientation $\mu_{S,\sigma}$: $\mathrm{Gr}_d^+(TB/\mathfrak{g}) \to \mathbb{Q}$ of $\lambda_{S,\sigma}$ such that

$$\mu_{S,\sigma}(x, F) = \sum_{\substack{s_j(x) = S(x) \\ \ker D(S - s_j)(x) = F}} \sigma_j - \sum_{\substack{s_j(x) = S(x) \\ \ker D(S - s_j)(x) = -F}} \sigma_j \tag{31}$$

for every collection of local branches (s_i, σ_i) of σ in an open set $U \subset B$ and every $x \in U$.

Proof. Consider the weighted branched submanifolds $\lambda_0, \lambda_1 : E \to \mathbb{Q}$ given by

$$\lambda_0(x,e) := \begin{cases} 1 & \text{if } e = 0, \\ 0 & \text{if } e \neq 0, \end{cases} \qquad \lambda_1(x,e) := \sigma(x, S(x) - e).$$

They correspond to the zero section and to the multivalued section $S - \sigma$, respectively. Then $\lambda_{S,\sigma}$ is just the intersection $\lambda_0 \lambda_1$, viewed as a weighted branched submanifold of B. So if B is oriented the result follows directly from (23). The nonoriented case can either be deduced from the oriented case by lifting S and σ to the bundle $E' \to B' := E$ whose fibre over (x,e) is $E_x \oplus E_x$ or be proved directly as follows.

First note that an isomorphism $\pi : F_0 \to F_1$ between two subspaces F_0, F_1 of an oriented vector space V such that $V = F_0 + F_1$ induces an orientation on $F_0 \cap F_1$: pick any orientations of F_0 and F_1 corresponding to each other under π and take the orientation induced on $F_0 \cap F_1$.

Since each branch of λ_1 is a section of E and is transverse to the zero section, every subspace

$$H \subset T_{(x,0)}E \cong T_xB \oplus E_x$$

such that $T\lambda_1((x,0), H) > 0$ satisfies

$$T_xB \times E_x = (T_xB \times 0) + H$$

and is isomorphic to T_xB under the projection $\mathrm{d}\pi : TE \to TB$. Hence the intersection $(T_xB \times 0) \cap H$ carries a natural orientation. With this understood, the following formula defines an orientation of $\lambda_{S,\sigma}$ which satisfies (31) for any collection of local branches:

$$\mu_{S,\sigma}(x,F) := \sum_{\substack{H \subset T_xB \oplus E_x \\ (T_xB\times 0)\cap H = F\times 0}} T\lambda_1((x,0),|H|) - \sum_{\substack{H \subset T_xB \oplus E_x \\ (T_xB\times 0)\cap H = -F\times 0}} T\lambda_1((x,0),|H|).$$

Here $\mathrm{Gr}_d^+(T_xB \oplus E_x/\mathfrak{g}) \to \mathrm{Gr}_d(T_xB \oplus E_x/\mathfrak{g}) : F \mapsto |F|$ is the map that forgets the orientation. □

10.3. Existence of transverse multivalued sections

The next proposition asserts the existence of a multivalued perturbation which is transverse to S and is supported in an arbitrarily small neighbourhood of the zero set of S. The proof shows that the perturbation can be chosen arbitrarily small in the C^ℓ-topology (on the branches).

Proposition 10.6. *Let (B,E,S) be a finite dimensional regular* G*-moduli problem and $Z \subset B$ be a* G*-invariant neighbourhood of $M = S^{-1}(0)$. Then there exists a multivalued section $\sigma : E \to \mathbb{Q}\cap[0,\infty)$ with the following properties*:

(i) *σ is transverse to S.*
(ii) *σ is supported in Z, i.e. $\sigma(x,0) = 1$ for every $x \in B \setminus Z$.*

Proof. The proof has two steps.

Step 1: *There exists a positive integer N and a function*

$$\sigma : E \times \mathbb{R}^N \to \mathbb{Q} : (x,e,y) \mapsto \sigma_y(x,e)$$

with the following properties:

(i) *σ is a multivalued section of the bundle $E \times \mathbb{R}^N \to B \times \mathbb{R}^N$ with respect to the diagonal action of* G, *where* G *acts trivially on $\mathbb{R}^N$.*
(ii) *σ is linear in y, i.e. $\sigma_{y_1+y_2} = \sigma_{y_1} * \sigma_{y_2}$ and $\sigma_{ty} = t\sigma_y$ for $y_1, y_2, y \in \mathbb{R}^N$ and $t \in \mathbb{R}$.*
(iii) *The multivalued section $\sigma_y : E \to \mathbb{Q}$ is supported in Z for every $y \in \mathbb{R}^N$, i.e. $\sigma_y(x,0) = 1$ for every $x \in B \setminus Z$ and every $y \in \mathbb{R}^N$.*
(iv) *For every local branch $s : V \times W \to E|_V$ of σ, defined on the product of two open sets $V \subset B$ and $W \subset \mathbb{R}^N$ with $0 \in W$, and every $x \in V \cap M$ the derivatives $\partial_{y_i} s(x,0)$, $i = 1,\dots,N$, span the vector space E_x.*

Given $x_0 \in M$ choose a local slice $(U_0, \phi_0, \mathrm{G}_0)$ of B/G such that

$$x_0 = \phi_0(0), \qquad \mathrm{G}^*\phi_0(U_0) \subset Z.$$

Let $E_0 := E_{x_0}$ and suppose that U_0 is a contractible neighbourhood of zero in (the finite dimensional G-Hilbert space) H_0. Then there exists a G_0-equivariant trivialization

$$U_0 \times E_0 \to \phi_0^* E : (x,v) \mapsto \Phi_x v \in E_{\phi_0(x)}.$$

Choose finitely many smooth functions $s_1,\dots,s_n:U_0 \to E_0$ with compact support such that the vectors $s_1(0),\dots,s_n(0)$ form a basis of E_0 and define $\sigma_0:E\times\mathbb{R}^n\to\mathbb{Q}$ by

$$\sigma_{0y}(g^*\phi_0(x),g^*e):=\frac{1}{|\mathrm{G}_0|}\left|\left\{g_0\in\mathrm{G}_0\,\middle|\,\sum_{i=1}^n y_i\Phi_x s_i(x)=g_0^*e\right\}\right|$$

for $x\in U_0$, $e\in E_{\phi_0(x)}$, $g\in\mathrm{G}$, and $y\in\mathbb{R}^n$ and by $\sigma_{0y}(x,0)=1$ for $x\in B\setminus\mathrm{G}^*\phi_0(U_0)$ and $y\in\mathbb{R}^n$. Then σ_0 satisfies (i)–(iii) and satisfies (iv) in a neighbourhood of x_0.

Now cover M by finitely many open sets $V_1,\dots,V_N$ such that, for each i, there exists a multivalued section $\sigma_i:E\times\mathbb{R}^n\to\mathbb{Q}$ which satisfies (i)–(iii) and satisfies (iv) in V_i. Then the multivalued section $\sigma:E\times\mathbb{R}^{nN}\to\mathbb{Q}$ defined by

$$\sigma_y:=\sigma_{1y_1}*\cdots*\sigma_{Ny_N}$$

for $y=(y_1,\dots,y_N)\in\mathbb{R}^{nN}$ satisfies the requirements of Step 1.

Step 2: *We prove the proposition.*

Let $\sigma:E\times\mathbb{R}^N\to\mathbb{Q}$ be as in Step 1. Then there exists a $\delta>0$ such that set

$$\mathscr{M}_\sigma:=\{(x,y)\in B\times\mathbb{R}^N\mid\sigma_y(x,S(x))>0,\ |y|<\delta\}$$

is (the support of) an oriented weighted branched $(d+N)$-submanifold of $B\times\mathbb{R}^N$. Let $y\in\mathbb{R}^N$ be a sufficiently small regular value of the obvious projection $\mathscr{M}_\sigma\to\mathbb{R}^N$. Then $\sigma_y:E\to\mathbb{Q}$ satisfies the requirements of the proposition. □

10.4. Multivalued classifying maps

If G acts freely on B then there is an equivariant classifying map $\theta:B\to\mathrm{EG}$, unique up to homotopy. In the presence of finite isotropy subgroups there is no such map. However, it is possible to construct an equivariant *multivalued* map $\Theta:B\to 2^{\mathrm{EG}}$ which assigns a finite subset $\Theta(x)\subset\mathrm{EG}$ to every point $x\in B$. Such a map gives rise to a branched submanifold of $B\times_{\mathrm{G}}\mathrm{EG}$ which in turn determines a rational cycle. Here is how this works.

Definition 10.7. Suppose G acts on the finite dimensional manifold B with finite isotropy. A *multivalued classifying map* on B is a multivalued section of the trivial bundle $B\times\mathrm{EG}\to B$. Explicitly, it is a function

$$\nu:B\times\mathrm{EG}\to\mathbb{Q}\cap[0,\infty)$$

with the following properties.

(**Equivariance**) $\nu(g^*x,g^{-1}\theta)=\nu(x,\theta)$ for all $x\in B$, $\theta\in\mathrm{EG}$, and $g\in\mathrm{G}$.

(**Local structure**) For every $x_0\in B$ there exist an open neighbourhood U, smooth functions $\theta_1,\dots,\theta_m:U\to\mathrm{EG}$, and positive rational numbers $\nu_1,\dots,\nu_m$ such that

$$\sum_{i=1}^m\nu_i=1,\qquad \nu(x,\theta)=\sum_{\theta_i(x)=\theta}\nu_i$$

for every $x\in U$ and every $e\in\mathrm{EG}$.

Remark 10.8. Let $v : B \times \mathrm{EG} \to \mathbb{Q} \cap [0, \infty)$ be a multivalued classifying map. Then, for every $x \in B$, the set

$$\Theta(x) := \{\theta \in \mathrm{EG} \mid v(x, \theta) > 0\}$$

is finite and

$$\sum_{\theta \in \Theta(x)} v(x, \theta) = 1.$$

Moreover, $\Theta(x) = \{\theta_1(x), \ldots, \theta_m(x)\}$, where the θ_i are the local branches of v.

Remark 10.9. A multivalued classifying map $v : B \times \mathrm{EG} \to \mathbb{Q}$ descends to a weighted branched submanifold of $B \times_{\mathrm{G}} \mathrm{EG}$.

Proposition 10.10. (i) *Every finite dimensional smooth* G*-manifold* B *with finite isotropy subgroups admits a multivalued classifying map.*
(ii) *Any two multivalued classifying maps are equivariantly homotopic.*

Proof. The proof of (i) has three steps. The proof of (ii) is similar and is left to the reader.

Step 1: *For every point* $x_0 \in B$ *there exist a* G*-invariant open neighbourhood* $U_0 \subset B$ *of* x_0, *a finite subgroup* $\mathrm{G}_0 \subset \mathrm{G}$, *and a set-valued function* $\Theta_0 : U_0 \to 2^{\mathrm{G}}$ *such that*

(i) $\Theta_0(x)$ *has* $|\mathrm{G}_0|$ *elements for every* $x \in U_0$.
(ii) $\Theta_0(g^* x) = \Theta_0(x) g$ *and* $\Theta_0(g_0^* x) = g_0^{-1} \Theta_0(x)$ *for all* $x \in U_0$, $g \in \mathrm{G}$, *and* $g_0 \in \mathrm{G}_0$.
(iii) *For every* $x \in U_0$ *there exist an open neighbourhood* $U \subset U_0$ *of* x *and smooth functions* $g_i : U \to \mathrm{G}$ *for* $i = 1, \ldots, m_0 := |\mathrm{G}_0|$ *such that* $\Theta_0(x) = \{g_1(x), \ldots, g_{m_0}(x)\}$ *for every* $x \in U$.

Step 1 follows directly from the local slice Theorem 4.1. Given a local slice $\phi_0 : W_0 \to B$ define $U_0 := \mathrm{G}^* \phi_0(W_0)$ and $\Theta_0(x) := \{g \in \mathrm{G} \mid x \in g^* \phi_0(W_0)\}$.

Step 2: *Assertion* (i) *holds when* B *can be covered by finitely many local slices.*

We may assume without loss of generality that $\mathrm{G} \subset U(k)$. Then a finite dimensional approximation of the space EG is given by

$$\mathrm{EG}^n := \{\theta \in \mathbb{C}^{k \times n} \mid \theta\theta^* = \mathbb{1}\}.$$

The group G acts on EG^n by $\theta \mapsto g\theta$ for $g \in \mathrm{G} \subset U(k)$.

Now cover B by finitely many G-invariant open sets $U_1, \ldots, U_N$ such that, for every $i \in \{1, \ldots, N\}$, there exist a finite subgroup $\mathrm{G}_i \subset \mathrm{G}$ and a set-valued function $\Theta_i : U_i \to 2^{\mathrm{G}}$ satisfying the requirements of Step 1. Pick G-invariant smooth cutoff functions $\rho_1, \ldots, \rho_N : B \to [0, 1]$ such that

$$\operatorname{supp} \rho_i \subset U_i, \quad \sum_{i=1}^{N} \rho_i^2 = 1.$$

Write a matrix $\theta \in \mathrm{EG}^{Nk}$ as a row of $(k \times k)$-blocks $\theta_1, \dots, \theta_N \in \mathbb{C}^{k\times k}$ such that $\sum_{i=1}^N \theta_i \theta_i^* = \mathbb{1}$. With this understood define $v : B \times \mathrm{EG}^{Nk} \to \mathbb{Q}$ by

$$v(x,\theta) := \prod_{i=1}^{N} \frac{|\{h \in \Theta_i(x) \mid \rho_i(x)h^* = \theta_i\}|}{|\mathrm{G}_i|}.$$

Then, for every $x \in B$, the set $\Theta(x) := \{\theta \in \mathrm{EG}^{Nk} \mid v(x,\theta) > 0\}$ consists of at most $\prod_i |\mathrm{G}_i|$ elements. The formula $v(g^*x, g^{-1}\theta) = v(x,\theta)$ follows from the fact that $\Theta_i(g^*x) = \Theta_i(x)g$. The formula $\sum_\theta v(x,\theta) = 1$ follows from the fact that the subset $\Theta_i(x) \subset \mathrm{G}$ consists of $|\mathrm{G}_i|$ elements. The (*Local structure*) axiom follows from (iii) in Step 1.

Step 3: *Assertion* (i) *holds in general.*

Since B is paracompact it admits a locally finite countable cover $\{U_i\}_i$ such that for each i there exist a finite subgroup $\mathrm{G}_i \subset \mathrm{G}$ and a setvalued function $\Theta_i : U_i \to 2^{\mathrm{G}}$ as in Step 1. Now choose a G-invariant partition of unity $\rho_i^2 : B \to [0,1]$ and repeat the construction of Step 2 with EG^{Nk} replaced by the infinite dimensional space $\mathrm{EG} = \bigcup_N \mathrm{EG}^{Nk}$. □

Corollary 10.11. *Let B be a smooth oriented finite dimensional manifold and* G *be a compact oriented Lie group which acts on B by orientation preserving diffeomorphisms and with finite isotropy. Suppose that $\lambda : B \to \mathbb{Q}$ is a* (G-*invariant*) *compact oriented weighted branched d-submanifold of B. Then there exists a rational homology class $[\lambda] \in H_d(B \times_{\mathrm{G}} \mathrm{EG}; \mathbb{Q})$ in singular homology such that*

$$\langle [\alpha], [\lambda] \rangle = \int_{\lambda/\mathrm{G}} \alpha$$

for every G-*closed equivariant differential form $\alpha \in \Omega_{\mathrm{G}}^d(B)$. Here we denote by $[\alpha] \in H^*(B \times_{\mathrm{G}} \mathrm{EG}; \mathbb{R})$ the equivariant cohomology class of α.*

Proof. Shrinking B, if necessary, we may assume that there exists a multivalued equivariant classifying map $v : B \times \mathrm{EG}^n \to \mathbb{Q}$ to a finite dimensional approximation of EG. Consider the compact oriented weighted branched d-submanifold $\lambda^n : B \times_{\mathrm{G}} \mathrm{EG}^n \to \mathbb{Q}$ defined by

$$\lambda^n([x,\theta]) := \lambda(x) v(x,\theta).$$

Geometrically, λ^n corresponds to the image of the support of λ under the multivalued classifying map v, divided by the free G-action on EG. By Theorem 9.23, there exists a rational homology class $[\lambda^n] \in H_d(B \times_{\mathrm{G}} \mathrm{EG}^n; \mathbb{Q})$ such that

$$\langle [\beta], [\lambda^n] \rangle = \int_{\lambda^n} \beta$$

for every closed form $\beta \in \Omega^d(B \times_{\mathrm{G}} \mathrm{EG}^n)$. Now let $\alpha \in \Omega_{\mathrm{G}}^d(B)$ be G-closed and $A \in \Omega^1(B, \mathfrak{g})$ be a connection 1-form. Then, by Theorem 3.8, α_A is a closed G-invariant horizontal d-form on B. The induced cohomology class in $H^d(B \times_{\mathrm{G}} \mathrm{EG}^n; \mathbb{R})$ is given by

$$[\alpha^n] := [\pi_B^* \alpha_A] \in H^d(B \times_{\mathrm{G}} \mathrm{EG}^n; \mathbb{R}),$$

where $\pi_B : B \times \mathrm{EG}^n \to B$ denotes the obvious projection. Note that $\pi_B^*\alpha_A$ is closed, G-invariant, and horizontal, and hence descends to a closed d-form on $B \times_{\mathrm{G}} \mathrm{EG}^n$, still denoted by $\pi_B^*\alpha_A$. We have

$$\langle [\alpha^n], [\lambda^n] \rangle = \langle [\pi_B^*\alpha_A], [\lambda^n] \rangle = \int_{\lambda^n} \pi_B^*\alpha_A = \int_{\lambda/\mathrm{G}} \alpha_A = \int_{\lambda/\mathrm{G}} \alpha.$$

Here the penultimate identity follows from Proposition 10.4. Note also that this formula shows that the cohomology class $[\lambda^n]$ is independent of the choice of v. The pushforward $[\lambda] \in H_d(B \times_{\mathrm{G}} \mathrm{EG}; \mathbb{Q})$ of $[\lambda^n]$ under the inclusion $B \times_{\mathrm{G}} \mathrm{EG}^n \to B \times_{\mathrm{G}} \mathrm{EG}$ satisfies the requirements of the corollary. □

10.5. Poincaré duality

The next theorem is a version of Poincaré duality. It asserts that the zero set of a transverse multivalued section is Poincaré dual to the pullback of the Thom class.

Theorem 10.12. *Let (B, E, S) be a finite dimensional regular* G*-moduli problem and (U, τ) be a Thom structure on (B, E, S). Let $d := \mathrm{index}(S)$ and $n := \mathrm{rank}\, E$. If $\sigma : E \to \mathbb{Q}$ is a multivalued section that is transverse to S and has compact support, then*

$$\int_{B/\mathrm{G}} \alpha \wedge S^*\tau = \int_{\lambda_{S,\sigma}/\mathrm{G}} \alpha \tag{32}$$

for every $\alpha \in \Omega^d_{\mathrm{G}}(B)$ such that $d_{\mathrm{G}}\alpha = 0$.

Proof. The proof has three steps.

Step 1: *The theorem holds in the case* $\mathrm{G} = \{\mathbb{1}\}$.

In this case (32) can be restated in the form

$$\int_B \alpha \wedge S^*\tau = \int_{\lambda_{S,\sigma}} \alpha. \tag{33}$$

This equation asserts that the closed compactly supported differential form $S^*\tau \in \Omega^*(B)$ is Poincaré dual to the homology class $[\lambda_{S,\sigma}]$. We claim that the class $[\lambda_{S,\sigma}]$ is equal to the rational homology class of $M_0 := S_0^{-1}(0)$, where $S_0 : B \to E$ is a smooth section which is transverse to the zero section and agrees with S outside of a compact set. To see this choose a regular homotopy from S_0 to $S - \sigma$. The zero set of such a homotopy is a branched submanifold with boundary $\{0\} \times M_0 \cup \{1\} \times \lambda_{S,\sigma}$ in $[0, 1] \times B$. It now follows from Proposition 9.21(ii) that

$$\int_{M_0} \alpha = \int_{\lambda_{S,\sigma}} \alpha$$

for every closed form $\alpha \in \Omega^d(B)$ and so $[M_0] = [\lambda_{S,\sigma}] \in H_d(B; \mathbb{Q})$ as claimed. With this understood equation (33) follows from [4, Proposition 12.8] (and also from Corollary 6.4 above).

Step 2: *Assume* $\mathrm{G} = \{\mathbb{1}\}$. *Then*

$$\int_{\lambda'} \alpha \wedge S^*\tau = \int_{\lambda_{S,\sigma}\lambda'} \alpha \tag{34}$$

for every oriented weighted branched d'-submanifold $\lambda' : B \to \mathbb{Q}$ that is transverse to $\lambda_{S,\sigma}$ and has closed support and every closed form $\alpha \in \Omega^{d+d'-\dim B}(B)$.

Eq. (33) asserts that the form $S^*\tau$ is Poincaré dual to the homology class $[\lambda_{S,\sigma}]$. Hence (34) follows from Step 7 in the proof of Theorem 9.23.

Step 3: *The theorem holds in general.*

Let EG^n be a finite dimensional approximation of EG and $v : B \times \mathrm{EG}^n \to \mathbb{Q}$ be a multivalued classifying map. Note that EG^n is a smooth compact manifold. Consider the vector bundle

$$\tilde{E} := E \times_{\mathrm{G}} \mathrm{EG}^n \to \tilde{B} := B \times_{\mathrm{G}} \mathrm{EG}^n.$$

The section $S : B \to E$ induces a section $\tilde{S} : \tilde{B} \to \tilde{E}$, given by

$$\tilde{S}([x,\theta]) := [x, S(x), \theta]$$

and the multivalued perturbation $\sigma : E \to \mathbb{Q}$ determines a compactly supported multivalued perturbation $\tilde{\sigma} : \tilde{E} \to \mathbb{Q}$ given by

$$\tilde{\sigma}([x,e,\theta]) := \sigma(x,e).$$

It follows from the hypotheses that $\tilde{\sigma}$ is transverse to $\tilde{S}$ and that the zero sets of both $\tilde{S}$ and $\tilde{S} - \tilde{\sigma}$ are compact. The latter is the compact oriented weighted branched submanifold $\tilde{\lambda} := \lambda_{\tilde{S},\tilde{\sigma}} : \tilde{B} \to \mathbb{Q}$ given by

$$\tilde{\lambda}([x,\theta]) := \sigma(x, S(x)).$$

We shall also abbreviate $\lambda := \lambda_{S,\sigma}$. The multivalued classifying map v descends to a weighted branched manifold $\tilde{v} : \tilde{B} \to \mathbb{Q}$, given by

$$\tilde{v}([x,\theta]) := v(x,\theta),$$

which is transverse to $\tilde{\lambda}$.

Now let $\tau \in \Omega^n(E)$ be a G-invariant and horizontal Thom form and $\alpha \in \Omega^d(B)$ be a closed G-invariant horizontal form. Denote by

$$\pi_B : B \times \mathrm{EG}^n \to B, \quad \pi_E : E \times \mathrm{EG}^n \to E$$

the obvious projections. Then $\pi_E^*\tau$ and $\pi_B^*\alpha$ are closed, G-invariant, and horizontal, and hence descend to closed forms on $E \times_{\mathrm{G}} \mathrm{EG}^n$ and $B \times_{\mathrm{G}} \mathrm{EG}^n$, which will be denoted by $\tilde{\tau}$ and $\tilde{\alpha}$, respectively. Note that $\tilde{\tau}$ is a Thom form for the bundle $\tilde{E} \to \tilde{B}$ and lifts to the G-invariant and horizontal form $\pi_B^* S^*\tau \in \Omega^*(B \times \mathrm{EG})$ under the obvious projection $B \times \mathrm{EG} \to B \times_{\mathrm{G}} \mathrm{EG}$. Since $v^*\pi_B^*\alpha = \alpha$ and $v^*\pi_B^* S^*\tau = S^*\tau$, it follows from Proposition 10.4 that

$$\begin{aligned}\int_{B/\mathrm{G}} \alpha \wedge S^*\tau &= \int_{v/\mathrm{G}} \pi_B^*(\alpha \wedge S^*\tau) \\ &= \int_{\tilde{v}} \tilde{\alpha} \wedge \tilde{S}^*\tilde{\tau} \\ &= \int_{\tilde{\lambda}\tilde{v}} \tilde{\alpha}\end{aligned}$$

$$= \int_{\lambda\nu/\mathrm{G}} \pi_B^* \alpha$$

$$= \int_{\lambda/\mathrm{G}} \alpha.$$

Here the first and fifth equalities follow from Proposition 10.4(iv), the second and fourth equalities follow directly from the definitions, and the third equality follows from Step 2. This proves the result for every G-invariant and horizontal closed d-form $\alpha \in \Omega^d(B)$. That the result continues to hold for every G-closed equivariant differential form $\alpha \in \Omega^*_{\mathrm{G}}(B)$ follows from Theorem 3.8. □

10.6. Rationality of the Euler class

Theorem 10.13. *Let $(\mathscr{B}, \mathscr{E}, \mathscr{S})$ be a regular* G*-moduli problem of index d. Then there exists a rational homology class $[\lambda] \in H_d(\mathscr{B} \times_{\mathrm{G}} \mathrm{EG}; \mathbb{Q})$ such that the homomorphism $\chi^{\mathscr{B},\mathscr{E},\mathscr{S}} : H^d(\mathscr{B} \times_{\mathrm{G}} \mathrm{EG}) \to \mathbb{R}$ is given by $\chi^{\mathscr{B},\mathscr{E},\mathscr{S}}(\alpha) = \langle \alpha, [\lambda] \rangle$.*

Corollary 10.14. *The Euler class satisfies the* (*Rationality*) *axiom.*

Proof of Theorem 10.13. By Theorem 7.4, it suffices to consider the finite dimensional case. Let (B, E, S) be a finite dimensional G-moduli problem and $\sigma : E \to \mathbb{Q}$ be a multivalued section transverse to S as in Proposition 10.6. Let $\lambda_{S,\sigma}$ be the oriented weighted branched d-submanifold of B defined in Proposition 10.5, where

$$d = \mathrm{index}(S) = \dim B - \mathrm{rank}\, E - \dim \mathrm{G}.$$

By Corollary 10.11, there exists a rational homology class

$$[\lambda_{S,\sigma}] \in H_d(B \times_{\mathrm{G}} \mathrm{EG}; \mathbb{Q})$$

such that

$$\langle [\alpha], [\lambda_{S,\sigma}] \rangle = \int_{\lambda_{S,\sigma}/G} \alpha = \int_{B/G} \alpha \wedge S^* \tau = \chi^{\mathscr{B},\mathscr{E},\mathscr{S}}(\alpha)$$

for every G-closed equivariant differential form $\alpha \in \Omega^d_{\mathrm{G}}(B)$. Here the second equality follows from Theorem 10.12 and the last one from the definition of the Euler class. □

11. Localization for circle actions

Let X be a compact connected oriented smooth manifold and

$$\mathscr{E}_\nu \to X, \quad \mathscr{F}_\nu \to X, \quad \nu = 1, \ldots, n$$

be complex Hilbert space bundles. For each ν let

$$\mathscr{D}_{\nu x} : \mathscr{E}_{\nu x} \to \mathscr{F}_{\nu x}$$

be a smooth family of complex linear Fredholm operators whose complex (numerical) index will be denoted by $\mathrm{index}(\mathscr{D}_v)$. Let us denote by

$$\mathrm{ind}(\mathscr{D}_v) := \bigcup_{x\in X} \{x\} \times \ker \mathscr{D}_{vx} \ominus \mathrm{coker}\, \mathscr{D}_{vx} \in K(X)$$

the topological index of $\mathscr{D}_v$ (as a K-theory class). Fix a sequence of nonzero integers $\ell = (\ell_1, \ldots, \ell_n)$ and consider the following S^1-moduli problem. The Hilbert manifold $\mathbb{B}$ is given by

$$\mathbb{B} := \left\{ (x, e_1, \ldots, e_n) | x \in X,\ e_v \in \mathscr{E}_{vx}, \sum_{v=1}^{n} \|e_v\|^2 = 1 \right\}$$

and the circle acts on $\mathbb{B}$ by

$$\lambda^*(x, e_1, \ldots, e_n) = (x, \lambda^{-\ell_1} e_1, \ldots, \lambda^{-\ell_n} e_n)$$

for $(x,e) \in \mathbb{B}$ and $\lambda \in S^1$. The Hilbert space bundle $\mathbb{H} \to \mathbb{B}$ has fibre

$$\mathbb{H}_{x,e} := \mathscr{F}_{1x} \oplus \cdots \oplus \mathscr{F}_{nx}$$

over $(x,e) \in \mathbb{B}$, and the section $\mathbb{S} : \mathbb{B} \to \mathbb{H}$ is given by

$$\mathbb{S}(x, e_1, \ldots, e_n) := (\mathscr{D}_{1x} e_1, \ldots, \mathscr{D}_{nx} e_n).$$

The zero set of this section is the *kernel manifold*

$$\mathbb{M} := \{(x, e_1, \ldots, e_n) \in \mathbb{B} \,|\, D_{vx} e_v = 0 \text{ for all } v\}.$$

Consider the action of S^1 on $\mathbb{B} \times \mathrm{E}S^1$ by $\lambda^*(x,e,\theta) = (t, \lambda^* e, \lambda^{-1}\theta)$, denote by $\pi_{\mathbb{B}} : \mathbb{B} \times_{S^1} \mathrm{E}S^1 \to \mathrm{B}S^1$ the projection, and let $c \in H^2(\mathrm{B}S^1; \mathbb{Z})$ be the positive generator. Recall that the *Chern series* of the K-theory class $\mathrm{ind}(\mathscr{D}) \in K(X)$ is defined by

$$c(\mathrm{ind}(\mathscr{D}), \eta) := \sum_{j \geqslant 0} \eta^{\mathrm{index}(\mathscr{D}) - j} c_j(\mathrm{ind}(\mathscr{D})),$$

where $\mathrm{index}(\mathscr{D}) := \dim\ker \mathscr{D} - \dim \mathrm{coker}\, \mathscr{D}$ is the Fredholm index. This series is multiplicative with respect to the Whitney sums. The following theorem can be interpreted as a localization formula: an invariant integral over the sphere bundle is expressed as an integral over the fixed point set X of the S^1-action.

Theorem 11.1. *Let k be a nonnegative integer and $\alpha \in H^{\dim X - 2k}(X)$. Suppose*

$$m + k - 1 \geqslant 0, \quad m := \sum_{v=1}^{n} \mathrm{index}(\mathscr{D}_v).$$

Then

$$\chi^{\mathbb{B},\mathbb{H},\mathbb{S}}(\pi_{\mathbb{B}}^* c^{m+k-1} \smile \pi^* \alpha) = \int_X \frac{\alpha}{\prod_{v=1}^{n} c(\mathrm{ind}(\mathscr{D}_v), \ell_v)}, \tag{35}$$

where $\pi : \mathbb{B} \times_{S^1} \mathrm{E}S^1 \to X$ denotes the projection.

Proof. The proof has three steps. The first is the case $X = \{\text{pt}\}$, $\mathscr{E}_v = \mathbb{C}$, $\mathscr{F}_v = \{0\}$, and $\alpha = 1$.

Step 1: *Suppose S^1 acts on $S^{2n-1} \subset \mathbb{C}^n$ by*

$$\lambda^*(z_1, \ldots, z_n) := (\lambda^{-\ell_1} z_1, \ldots, \lambda^{-\ell_n} z_n)$$

and let $\pi : S^{2n-1} \times_{S^1} \mathrm{E}S^1 \to \mathrm{B}S^1$ denote the projection. Then

$$\int_{S^{2n-1}/S^1} \pi^* c^{n-1} = \frac{1}{\ell_1 \cdots \ell_n}. \tag{36}$$

Consider the S^1-moduli problem

$$B := S^{2n-1}, \quad E := S^{2n-1} \times \mathbb{C}^{n-1}, \quad S(z) = (z_1, \ldots, z_{n-1}),$$

where S^1 acts on E by

$$\lambda^*(z, \zeta) := (\lambda^{-\ell_1} z_1, \ldots, \lambda^{-\ell_n} z_n, \lambda^{-\ell_1} \zeta_1, \ldots, \lambda^{-\ell_{n-1}} \zeta_{n-1}).$$

Let $\tau \in \Omega^{2n-2}(E)$ be an S^1-invariant horizontal Thom form. Then

$$[S^*\tau] = c_{n-1}(E \times_{S^1} \mathrm{E}S^1) = \ell_1 \cdots \ell_{n-1} \pi^* c^{n-1}.$$

Hence

$$\ell_1 \cdots \ell_{n-1} \int_{S^{2n-1}/S^1} \pi^* c^{n-1} = \chi^{B,E,S}(1) = \frac{1}{\ell_n}.$$

To prove the last equality note that S is transverse to the zero section. Its zero set is a single orbit with isotropy subgroup $\mathbb{Z}/\ell_n\mathbb{Z} \subset S^1$. Hence the equality follows from the (*Transversality*) axiom for the Euler class.

Step 2: *We may assume without loss of generality that $\mathscr{E}_v$ and $\mathscr{F}_v$ are finite dimensional and that each bundle $\mathscr{E}_v$ admits a trivialization.*

By Theorem 7.4 (in the nonequivariant case of complex Hilbert space bundles), there exists, for every v, a finite dimensional subbundle $F_v \subset \mathscr{F}_v$ such that

$$F_{vx} + \operatorname{im} \mathscr{D}_{vx} = \mathscr{F}_v$$

for every $x \in X$. Here we use the fact that, by a general position argument, we can choose the family of complements to be an embedding. Then the set

$$E_v := \{(x, e) \mid x \in X,\ e \in \mathscr{E}_v, \mathscr{D}_{vx} e \in F_v\}$$

is a subbundle of $\mathscr{E}_v$ and

$$\operatorname{rank} E_v - \operatorname{rank} F_v = \operatorname{index}(\mathscr{D}_v).$$

Let $D_v : E_v \to F_v$ denote the restriction of $\mathscr{D}_v$ to E_v. Then the S^1-moduli problem associated to the operators D_v admits an obvious morphism to $(\mathbb{B}, \mathbb{H}, \mathbb{S})$. Moreover, the right-hand side of (35) remains unchanged if we replace $\mathscr{D}_v$ by D_v. Hence, by the (*Functoriality*) axiom for the Euler class, we may assume that $\mathscr{E}_v = E_v$ and $\mathscr{F}_v = F_v$ are finite dimensional. In this case $\mathbb{B}$ is a compact smooth manifold and identity (35) has the form

$$\int_{\mathbb{B}/S^1} \pi_{\mathbb{B}}^* c^{m+k-1} \smile \pi^*\alpha \smile \mathbb{S}^*\tau = \int_X \frac{\alpha \smile \prod_{v=1}^n c(F_v, \ell_v)}{\prod_{v=1}^n c(E_v, \ell_v)}, \tag{37}$$

where $\tau \in \Omega^*(\mathbb{H})$ is an S^1-invariant horizontal Thom form. For each v there exists a complex vector bundle $E_v' \to X$ such that $E_v \oplus E_v'$ admits a trivialization. By the (*Functoriality*) axiom for the Euler class, the left-hand side of (37) remains unchanged if we replace E_v by $E_v \oplus E_v'$ and F_v by $F_v \oplus E_v'$. The right-hand side also remains unchanged under this operation and so we may assume without loss of generality that each bundle E_v admits a trivialization.

Step 3: *We prove the theorem.*

By Step 2, we may assume that $\mathscr{E}_v = E_v$ and $\mathscr{F}_v = F_v$ are finite dimensional and

$$E_v = X \times \mathbb{C}^{\operatorname{rank} E_v}$$

for every v. Then Eq. (37) has the form

$$\int_{\mathbb{B}/S^1} \pi_{\mathbb{B}}^* c^{m+k-1} \smile \pi^*\alpha \smile \mathbb{S}^*\tau = \prod_{v=1}^n \ell_v^{-\operatorname{rank} E_v} \int_X \alpha \smile \prod_{v=1}^n c(F_v, \ell_v). \tag{38}$$

Now we may assume that $D_v = 0$ for all v and hence $\mathbb{S}$ is the zero section. Let $\tau_v \in \Omega_{S^1}^{\operatorname{rank} F_v}(X)$ be the pullback under the zero section of an S^1-equivariant Thom form on F_v. Thus $\tau_v : \mathrm{i}\mathbb{R} \to \Omega^*(X)$ is a polynomial map whose coefficients are closed forms on X. Indeed, by Corollary 6.5,

$$\tau_v(\eta) = \sum_{j=0}^{\operatorname{rank} F_v} \left(\frac{\mathrm{i}\ell_v\eta}{2\pi}\right)^{\operatorname{rank} F_v - j} \tau_{vj}, \quad [\tau_{vj}] = c_j(F_v).$$

Since $\mathbb{S} : \mathbb{B} \to \mathbb{H}$ is the composition of the projection $\pi : \mathbb{B} \to X$ with the inclusion of the zero section into $F = F_1 \oplus \cdots \oplus F_n$, we have

$$\mathbb{S}^*\tau(\eta) = \prod_{v=1}^n \pi^*\tau_v(\eta) = \prod_{v=1}^n \left(\sum_{j=0}^{\operatorname{rank} F_v} \left(\frac{\mathrm{i}\ell_v\eta}{2\pi}\right)^{\operatorname{rank} F_v - j} \pi^*\tau_{vj} \right).$$

Since $\mathrm{i}\eta/2\pi$ represents the equivariant cohomology class $\pi_{\mathbb{B}}^* c \in H^2(\mathbb{B} \times_{S^1} \mathrm{E}S^1)$ (see Example 3.12), the cohomology class of $\mathbb{S}^*\tau$ is

$$[\mathbb{S}^*\tau] = \prod_{v=1}^n \left(\sum_{j=0}^{\operatorname{rank} F_v} (\ell_v \pi_{\mathbb{B}}^* c)^{\operatorname{rank} F_v - j} \smile \pi^* c_j(F_v) \right).$$

Hence Eq. (38) reads

$$\begin{aligned} &\int_{\mathbb{B}/S^1} \pi_{\mathbb{B}}^* c^{N-1} \smile \pi^* \left(\alpha \smile \prod_{v=1}^n \left(\sum_{j=0}^{\operatorname{rank} F_v} \ell_v^{\operatorname{rank} F_v - j} c_j(F_v) \right) \right) \\ &\quad = \prod_{v=1}^n \ell_v^{-\operatorname{rank} E_v} \int_X \alpha \smile \prod_{v=1}^n c(F_v, \ell_v), \end{aligned} \tag{39}$$

where $N := \sum_{\nu=1}^{n} \operatorname{rank} E_\nu$. Here we have used the fact that E_ν is the trivial bundle and so any power of $\pi_{\mathbb{B}}^* c$ that is higher than $N-1$ vanishes. Again, since E_ν is a trivial bundle, it follows from Step 1 that

$$\int_{\mathbb{B}/S^1} \pi_{\mathbb{B}}^* c^{N-1} \smile \pi^* \beta = \prod_{\nu=1}^{n} \ell_\nu^{-\operatorname{rank} E_\nu} \int_X \beta$$

for every $\beta \in H^{\dim X}(X)$. This implies (39) and completes the proof of the theorem. □

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